Theories of Iterated Positive Induction

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In 1963 Kreisel introduced the formal theories of inductive definitions in both single and iterated form.

In the 1970s these theories were studied in more details. This work culminated in [1].

Definition

Let $A(x, P^+)$ be an arithmetic formula with at most the free number variable x and a predicate P which occurs only positively in $A(x, P)$. Such a formula gives rise to a function $\Gamma_A : P(\mathbf{N}) \to P(\mathbf{N})$ via

$$\Gamma_A(Y) = \{n \in \mathbf{N} | A(n, Y)\}.$$

Γ_A is monotone, i.e.

$$Y \subseteq Z \implies \Gamma_A(Y) \subseteq \Gamma_A(Z).$$

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We can iterate Γ_A along the ordinals by defining

$$\Gamma_A^\alpha = \Gamma_A(\bigcup_{\beta<\alpha} \Gamma_A^\beta)$$

one creates the least fixed point I_A of Γ_A, i.e.

$$I_A = \bigcup_\alpha \Gamma_A^\alpha.$$

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A first order theory that formalizes these first order arithmetic inductive definitions is ID_1.

The language of ID_1 extends the language of PRA.
For each positive arithmetic formula $A(x, P^+)$, ID_1 has a unary predicate symbol I_A.

The axioms of ID_1 are those of PRA plus the induction scheme $IND_{\mathbf{N}}$:

$$F(0) \wedge \forall x[F(x) \rightarrow F(x+1)] \rightarrow \forall x F(x)$$

for all formulas $F(x)$ of ID_1.

In addition we have axioms for the predicates I_A:
(I_A.1) $\forall u[A(u, I_A) \rightarrow I_A(u)]$
(I_A.2) $\forall u[A(u, F) \rightarrow F(u)] \rightarrow \forall u[I_A(u) \rightarrow F(u)]$

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$I\hat{D}_1$

You can show

$$ID_1 \vdash \forall x(A(x, I_A) \leftrightarrow I_A(x)) \,.$$

$I\hat{D}_1$ is obtained from ID_1 by omitting the axioms (I_A.2) and adding the axioms

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(Aczel)

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The theory ID_1^* is obtained from ID_1 by restricting the scheme $I_A.2$ and $IND_{\mathbf{N}}$ to formulas $F(x)$ in which all predicates occur positively.

Friedman studied this system in 1969 as did Feferman in 1982, but only special cases were solved.

What was already known was that $\varphi\omega 0 \preceq |ID_1^*| \preceq \varphi\varepsilon 0$.

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Definition

The Σ_1^1-DC -Dependent Choices- scheme is

$$\forall x \forall X \exists Y B(x, X, Y) \to \forall U \exists Z[(Z)_0 = U \wedge \forall x B(x, (Z)_x, (Z)_{x+1})]$$

for Σ_1^1 formulas B.

The system $\Sigma_1^1 - DC_0$ is $ACA_0 + \Sigma_1^1 - DC$.

We can interpret ID_1^* in $\Sigma_1^1 - DC_0$; translate $I_A(t)$ using

$$\forall X[\forall u(A(u, X) \to u \in X) \to t \in X]$$

and leave anything else unchanged. For B a formula of ID_1^* we will denote the translated formula by B^*.

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More precisely

- $(\forall x\, A(x))^* \equiv \forall x\, A^*(x)$,
- $(\neg A)^* \equiv \neg A^*$,
- $(A \wedge B)^* \equiv A^* \wedge B^*$.,
- $(\forall X\, A(X))^* \equiv \forall X\, A^*(X)$.

Definition

A formula is essentially Π_1^1 if it belongs to the smallest collection of formulas which contains all arithmetical formulas and is closed under $\wedge, \vee, \exists x$, $\forall x$, and $\forall X$.

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2. $(A(x, I_A))^*$ *is essentially* Π_1^1

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For any essentially Π_1^1 formula G we can find a Π_1^1 formula G' with the same free variables such that

1. $\Sigma_1^1 - AC_0 \vdash G' \rightarrow G$
2. $ACA_0 \vdash G \rightarrow G'$

Theorem

(Simpson 1982) The following are equivalent over ACA_0:

1. $\Sigma_1^1 - DC$
2. *ω-model reflection for Π_2^1 formulas, i.e. if $C(X_1, \ldots, X_k)$ is Π_2^1-formula with all set parameters exhibited, then*

$$\begin{aligned} C(X_1, \ldots, X_k) \rightarrow \exists \mathbf{A}[X_1, \ldots, X_k \in \mathbf{A} \\ \mathbf{A} \models ACA_0 \\ \mathbf{A} \models C(X_1, \ldots, X_k)]. \end{aligned}$$

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Lemma

$\Sigma_1^1 - DC$ proves

$$\forall x[A(x,F) \rightarrow F(x)] \rightarrow \forall x[I_A^*(x) \rightarrow F(x)]$$

for all essentially Π_1^1 formulas $F(x)$.

Proof.

For a contradiction assume

(1) $\forall x[A(x,F) \rightarrow F(x)]$ but

(2) $I_A^*(n_0) \wedge \neg F(n_0)$ for some n_0

Let $G(x) := A(x,F)$. This formula is essentially Π_1^1. Let $G'(x)$ and $F'(x)$ be the corresponding formulas provided by the previous Lemma. Then we have

(3) $\forall x[G'(x) \rightarrow F'(x)]$ and

(4) $\neg F'(n_0)$.

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Proof.

Using Π_2^1 ω-model reflection there exists an ω-model $\mathbf{A}$ of ACA_0 such that

(5) $\mathbf{A} \models \forall x[G'(x) \rightarrow F'(x)]$ and

(6) $\mathbf{A} \models \neg F'(n_0)$

Second part of the Lemma implies

(7) $\mathbf{A} \models \forall x[A(x, F') \rightarrow G'(x)]$
and 5 and 7 together yield

(8) $\mathbf{A} \models \forall x[A(x, F') \rightarrow F'(x)]$.

Define $Z = \{u | \mathbf{A} \models F'(u)\}$. Z exists by arithmetical comprehension. As a result of 8 we have

(9) $\forall x[A(x, Z) \rightarrow x \in Z]$ and hence $I_A^* \subseteq Z$, thus by 2 $n_0 \in Z$, and therefore

(10) $\mathbf{A} \models F'(n_0)$.

6 and 10 are contradictory. □

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Define $Z = \{u | \mathbf{A} \models F'(u)\}$. Z exists by arithmetical comprehension. As a result of 8 we have

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The Strength of ID_1^*

Theorem

$$ID_1^* \vdash B \Rightarrow \Sigma_1^1\text{-}DC_0 \vdash B^*$$

Proof.

- $(I_A.1)^*$ is provable in ACA_0.
- $(I_A.2)^*$ is provable in $\Sigma_1^1 - DC_0$ using ω-model reflection for Π_2^1-formulas.
- Σ_1^1-DC_0 proves induction on $\mathbf{N}$ for ess-Π_1^1 formulas.

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Corollary

(Michael Rathjen 2007)

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Definition

The language of the subsystem ID_2^* *extends the language of* ID_1^**. In addition it has predicate symbols* I_A^2 *for formulas* $A(x, P^+)$ *in the language of* $ID_1^* \cup \{P\}$.

The axioms of ID_2^* *consist of the axioms from* ID_1^* *plus*

$$(I_A^2.1) \quad \forall u[A(u, I_A^2) \to I_A^2(u)]$$
$$(I_A^2.2) \quad \forall u[A(u, F) \to F(u)] \to \forall u[I_A^2(u) \to F(u)]$$

if all predicates of the form I_B^2 *appear positively in* F.

Definition

Let $X \subseteq P(\mathbf{N})$, $X \neq \emptyset$, and $\mathcal{A} = \langle \mathbf{N}, X, +, ., 0, 1, <, \in \rangle$.
$\mathcal{A}$ can be viewed as a structure for the language of second order arithmetic, where numerical quantifiers range over $\mathbf{N}$ and set quantifiers range over X.
If $X = \{(W)_n | n \in \mathbf{N}\}$ for some set $W \subseteq \mathbf{N}$, $\mathcal{A}$ is called a countable coded ω-model.

Definition

Let E_1, E_2,... be set constants. Let C_n ($N \geq 1$) express that $\langle \mathbf{N}, \{(E_n)_j | j \in \mathbf{N}\} \rangle$ is a model of $\Sigma_1^1 - DC_0$ and $E_1 \in \ldots \in E_{n-1} \in E_n$.

We can interpret ID_2^* in $\Sigma_1^1 - DC_0 + C_1$. The way to do this is to use the same interpretation from ID_1^* embedding but in two steps:

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We can interpret ID^*_2 in $\Sigma^1_1 - DC_0 + C_1$. The way to do this is to use the same interpretation from ID^*_1 embedding but in two steps:

(1) We translate the level one predicates and formulas of ID_1^* using the translation

$$(I_A^1(t))^{*E_1} \equiv \forall X \in E_1[\forall u(A(u,X) \to u \in X) \to t \in X].$$

(2) We then take the translation upward to predicates of level two by

$$(I_B^2(t))^{*E_1} \equiv \forall X[\forall u(B^{*E_1}(u,X) \to u \in X) \to t \in X].$$

Theorem

$$ID_2^* \vdash \psi \Rightarrow \Sigma_1^1 - DC_0 \vdash C_1 \to \psi^{*E_1}$$

where ψ^{*E_1} is the translation of ψ using the ω-model E_1.

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Theorem

$$ID^*_{n+1} \vdash \psi \Rightarrow \Sigma_1^1 - DC_0 \vdash \bigwedge_1^n C_i \to \psi^{*\vec{E}_n}$$

where $\psi^{*\vec{E}_n}$ *is the translation of* ψ *using the* ω*-models* $E_1, \ldots, E_n$.

Definition

For a limit ordinal $\alpha \preceq \Gamma_0$, *let* $(\Pi^0_1 - CA)_{\prec\alpha}$ *be the theory* ACA_0 *plus* $\forall X \exists Y H^X(\overline{\beta}, Y)$ *for all* $\beta \prec \alpha$ *and* $TI(\prec \alpha)$ *where* $H^X(\alpha, Y)$ *is defined as follows:*

$$\begin{aligned} H^X(\alpha, Y) \Leftrightarrow & (Y)_0 = X \wedge \\ & \forall \beta + 1 \preceq \alpha (Y)_{\beta+1} = jump((Y)_\beta) \wedge \\ & \forall \lambda \preceq \alpha (Y)_\lambda = \{\langle \xi, a \rangle | \xi \prec \lambda, a \in (Y)_\xi\}. \end{aligned}$$

Theorem

Let α *be an* ε*-number. If* Γ *is a set of ess-*Σ^1_1 *sentences, then*

$$\begin{aligned} & \Sigma^1_1 - DC_0 + TI(\prec \alpha) + C_1 + C_2 + \ldots + C_n \vdash \Gamma \\ & \qquad \Rightarrow (\Pi^0_1 - CA)_{\prec\alpha} + C_1 + C_2 + \ldots + C_n \vdash \Gamma . \end{aligned}$$

Definition

For a limit ordinal $\alpha \preceq \Gamma_0$, *let* $(\Pi^0_1 - CA)_{\prec\alpha}$ *be the theory* ACA_0 *plus* $\forall X \exists Y H^X(\overline{\beta}, Y)$ *for all* $\beta \prec \alpha$ *and* $TI(\prec \alpha)$ *where* $H^X(\alpha, Y)$ *is defined as follows:*

$$
\begin{aligned}
H^X(\alpha, Y) \Leftrightarrow & (Y)_0 = X \wedge \\
& \forall \beta + 1 \preceq \alpha (Y)_{\beta+1} = jump((Y)_\beta) \wedge \\
& \forall \lambda \preceq \alpha (Y)_\lambda = \{\langle \xi, a \rangle | \xi \prec \lambda, a \in (Y)_\xi\}.
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Lemma

If A is an ess-Π^1_1 formula then

$$RA^* \vdash^{\prec \omega.(\sigma+1).3}_{0} \neg A^\sigma, A^1 \, .$$

Theorem

Let λ be a limit and Γ a set of ess-Π^1_1 formulas. Then

$$(\Pi^0_1 - CA)_{\prec \omega^\lambda} + C_1 + \ldots + C_n \vdash \Gamma \Rightarrow RA^* + C_1 + \ldots + C_n \vdash^{\omega^{\lambda+1}}_{\prec \omega^\lambda} \Gamma^1$$

where Γ^1 arises from Γ by replacing every universal quantifier $\forall X$ by $\forall X^0$.

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Theorem

Let $RA^* + C_1 + \ldots + C_n \vdash^{\beta}_{q} \Gamma$ *mean that there is a derivation of length* $\preceq \beta$ *of* Γ *in* RA^* *where are cuts have cut formulas arithmetic in* $E_1, \ldots, E_n$*. Then*

$$RA^* + C_1 + \ldots + C_n \vdash^{\beta}_{\omega^\gamma} \Gamma \Rightarrow RA^* + C_1 + \ldots + C_n \vdash^{\varphi\gamma\beta}_{q} \Gamma .$$

Theorem

Let α *be an* ε*-number and* F *a formula of second order arithmetic relativized to* E_n*, then*

$$RA^* + C_1 + \ldots + C_n \vdash^{\prec\alpha}_{q} F^{E_n} \Rightarrow \Sigma^1_1 - DC_0 + C_1 + \ldots + C_{n-1} + TI(\prec \alpha) \vdash F^{\#} .$$

where $\#$ *is a translation which works in the following way*

- $(E_i \in E_n)^{\#} \equiv 0 = 0$ *where by* $E_i \in E_n$ *we mean* $\exists x\, (E_n)_x = E_i$
- $E_n^{\#} = \{x | x = x\}$.

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Corollary

Let A be an arithmetic formula, then

$$\Sigma_1^1 - DC + C_1 \vdash A \Rightarrow (\Pi_1^0 - CA)_{\prec\varphi\omega 0} \vdash A\,.$$

Proof.

As in [3], $\Sigma_1^1 - DC + C_1 \vdash A$ implies $(\Pi_1^0 - CA)_{\prec\omega^\omega} + C_1 \vdash A$. Then we would have $RA^* + C_1 \frac{\omega^{\omega+1}}{\omega^n} A$ for some n, and thus $RA^* + C_1 \frac{\varphi n\omega^{\omega+1}}{q} A$. From this we can derive $\Sigma_1^1 - DC_0 + TI(\prec \varphi n\omega^{\omega+1}) \vdash A$. Finally we get $(\Pi_1^0 - CA)_{\prec\varphi\omega 0} \vdash A$. □

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Corollary

ID_2^*, $\Sigma_1^1 - DC_0 + C_1$, and $(\Pi_1^0 - CA)_{\prec\varphi\omega 0}$ prove the same arithmetic statements as $PA + TI(\prec \varphi(\varphi\omega 0)0)$.

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$$|ID_n^*| = \underbrace{\varphi \ldots \varphi(\varphi}_{n}\omega 0)0 \ldots 0.$$

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