Polar decomposition of o-minimal groups

From not compact to compact

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o-minimal structures

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Real closed fields.

$\mathbb{R}_{\exp} = \langle \mathbb{R}, <, +, \cdot, \exp \rangle$ [Wilkie].

We fix a sufficiently saturated o-minimal expansion $\mathcal{M}$ of a real closed field.

Definable groups - Compact Lie groups

Theorem. [Berarducci - Otero - Peterzil - Pillay]
Every definable group G has a smallest type-definable subgroup of bounded index G^{00} and the quotient G/G^{00} with the logic topology is a compact real Lie group.

Theorem. [Hrushovski - Peterzil - Pillay] *If G is definably compact then $\dim G = \dim(G/G^{00})$.*

Theorem. [Hrushovski - Pillay] *If G is definably compact then G is dominated by $(G/G^{00}, h)$ under the canonical $\pi \colon G \to G/G^{00}$, where h is the Haar measure on G/G^{00}.*

Question. What is the meaning of G/G^{00} when G is not definably compact? Is G/G^{00} a 'good' compact Lie group in order to study topological properties of G?

G^{00}

G^{00} := the smallest type-definable subgroup of bounded index in G.

- Compact case: G^{00} is the subgroup of the 'intrinsecal' infinitesimals of G.

  Example. $G = [0,1[^M$, $G^{00} = \bigcap_{n\in\mathbb{N}} [0,1/n[^M$, $G/G^{00} \cong [0,1[^{\mathbb{R}} \cong S^1$.

- Non compact case: It can happens $G = G^{00}$:
  - G torsion free. Ex: $G = (M, +)$.
  - G almost definably simple. Ex: $G = SL(2, M)$.

Definable polar decomposition

Definition. A group G defined over a real closed field M has a definable polar decomposition if

- $\exists\, K < G$ maximal definably compact definable subgroup,
- $\exists\, E \subset G$, $E \approx M^l$,

s.t. the map

$$K \times E \to G$$

$$(k, e) \mapsto ke$$

is a definable homeomorphism.

$G \overset{?}{\approx} K \times M^l$

A negative example. [Strzebonski].

$G = \mathbb{R} \times [0, 1[$

$$(a_1, t_1) * (a_2, t_2) = \begin{cases} (a_1 + a_2, \qquad t_1 \oplus t_2) & \text{if } t_1 + t_2 < 1, \\ (a_1 + a_2 + 1, \ t_1 \oplus t_2) & \text{otherwise.} \end{cases}$$

$G \not\approx \mathbb{R}^2$ BUT it does not have any infinite definably compact definable subgroups.

$G \approx K \times M^l$

Proposition. [C.] *For every definable group G there exists a normal torsion free definable subgroup H of G which contains every normal torsion free definable subgroup of G.* We will say that it is the maximum normal torsion free definable subgroup of G.

Theorem. [C.] *Let $H \lhd G$ be the maximum normal torsion free definable subgroup of G. Then $\bar{G} := G/H$ does have a definable polar decomposition*:

$$\bar{G} \; \approx \; \bar{K} \times M^l.$$

Corollary. *Every definable group G is definably homotopically equivalent to a definably compact definable group $\bar{K}$.*

The functor: $G \mapsto G/G^{00}$

[Berarducci]

$$\Phi\colon \quad A \xrightarrow{f} B \qquad \mapsto \qquad A/A^{00} \xrightarrow{\Phi(f)} B/B^{00}$$

$$aA^{00} \mapsto f(a)B^{00}$$

The image of a short exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$$

is a short exact sequence

$$1 \longrightarrow N/N^{00} \xrightarrow{\Phi(i)} G/G^{00} \xrightarrow{\Phi(p)} Q/Q^{00} \longrightarrow 1$$

if and only if $N^{00} = N \cap G^{00}$.

Exactness property

Definition. G has the exactness property if

$$N \lhd G \text{ definable } \Rightarrow N^{00} = G^{00} \cap N.$$

G has the strong exactness property if

$$S < G \text{ definable } \Rightarrow S^{00} = G^{00} \cap S.$$

Negative examples.

- $G = SL(2, M) = G^{00}$, $S = SO(2, M)$, $N = \pm I$.
- $PSL(2, M) = SL(2, M)/\{\pm I\}$ does have the exactness property because is definably simple but not the strong because of $S = PSO(2, M)$.

$N \lhd G \overset{?}{\Rightarrow} N^{00} = G^{00} \cap N$

Proposition. [C.] *Let $H \lhd G$ definable groups s.t.*

- H *has the (strong) exactness property,*
- G/H *has the (strong) exactness property,*
- $H^{00} = H \cap G^{00}$.

Then G does have the (strong) exactness property.

Positive examples.

- strong:
  G with radical R s.t. G/R is definably compact.
- maybe not strong:
  G semisimple centerless.

Definable groups - Compact Lie groups

Let G be a definable group and $\bar{K}$ definably compact, definably homotopically equivalent to G:

$$\begin{tikzcd} G \arrow[r, phantom, "\sim"] \arrow[d, "p_G"'] & \bar{K} \arrow[d, "p_{\bar{K}}"] \\ G/G^{00} \arrow[r, leftrightarrow, "?"] & \bar{K}/\bar{K}^{00} \end{tikzcd}$$

In general G/G^{00} is Lie isomorphic to a quotient of $\bar{K}/\bar{K}^{00}$.

$G/G^{00} \cong \bar{K}/\bar{K}^{00} \iff G$ has the strong exactness property.

Two interesting subclasses

Let be R the radical of a definable group G. We have:

- R definably compact $\implies G \approx K \times M^l$, $K < G$

- G/R definably compact $\implies G$ has the strong exactness property

$\iff G/G^{00}$ is a 'good' compact real Lie group for G:

$$\Phi \colon G \mapsto G/G^{00} \text{ is exact and}$$

$$G/G^{00} \cong \bar{K}/\bar{K}^{00}$$