Degree Spectra of Almost Computable Structures

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A graph is computable if the edge relation is computable

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Theorem (Julia Knight)

The degree spectrum of a non-trivial structure is upward closed in the Turing degrees.

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By the $0-1$-Kolmogorov Law a structure is almost computable if and only if there is some $e \in \omega$ with $\mu(C_{\Phi_e}(\mathcal{M})) > 0$.

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Question

Is there an almost computable structure $\mathcal{M}$ for which the set $2^\omega - C(\mathcal{M})$ is uncountable?

Immunity Properties

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An infinite set X is hyperhyperimmune if there is no computable function f such that $\{W_{f(n)}\}_{n\in\omega}$ is a disjoint weak array and for all $n \in \omega$, $W_{f(n)} \cap X \neq \emptyset$.

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Theorem (Kuznecov, Medvedev, Uspenskii)

An infinite set X is hyperimmune if and only if no computable function dominates p_X.

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The characterization of hyperimmune sets passes to degrees, in the sense that a degree $\mathbf{d}$ is hyperimmune if and only if there exists a $\mathbf{d}$-computable function that is not dominated by any computable function.

Upward Closure

A general theorem of Jockusch shows that the immune, hyperimmune, and hyperhyperimmune degrees are all upward closed in the Turing degrees.

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The proof hinges on the fact that any infinite subset of an immune (hyperimmune, hyperhyperimmune) set is also immune (hyperimmune, hyperhyperimmune).

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There exists a degree $\mathbf{d}$ *that is immune but not bi-immune.*

Theorem

All hyperimmune degrees are bi-hyperimmune.

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The bi-immune degrees are upward closed in the Turing degrees.

Since the bi-hyperimmune degrees correspond to the hyperimmune degrees, they are upward closed in the Turing degrees.

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Definition

For any $e \in \omega$ and any set $X \subset \omega$, let $X^{[e]} = \{x \mid \langle e, x \rangle \in X\}$.

Wehner's Example

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Let $\mathcal{J} = \{\{n\} \oplus F \mid |F| < \infty \wedge F \neq W_n\}$

Then $\mathbf{Sp}(\mathcal{J}) = \mathbf{D} - \{\mathbf{0}\}$.

Proposition

There is an X-computable copy of $\mathcal{J}$ if and only if there exists a set $Y \equiv_T X$ such that $(\forall e)[|Y^{[e]}| < \infty]$ and $(\forall e)[Y^{[e]} \neq W_e]$.

An Immune-like Example

Let $\mathcal{F} = \{\{n\} \oplus F \mid |F| < \infty \wedge (|W_n| = \infty \rightarrow W_n \cap F \neq \emptyset)\}$.

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Note

For any e, s, we can compute $W_{g(e,s)}$ such that if $|W_e| = \infty$ then $|W_{g(e,s)}| = \infty$, $W_{g(e,s)} \cap \{0, \ldots, s\} = \emptyset$, and $W_{g(e,s)} \subseteq W_e$.

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If X has hyperimmune degree, then there is an X-computable copy of $\mathcal{F}$.

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If there is an X-computable copy of $\mathcal{F}$ then X has bi-immune degree.

Corollary

The graph $\mathcal{F}$ is almost computable but the class of degrees $\mathbf{D} - \mathbf{Sp}(\mathcal{F})$ is uncountable.

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Martin (unpublished) showed that the measure of the members of hyperimmune degrees is equal to one. So $\mathcal{F}$ is almost computable.

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Jockusch proved the existence of bi-immune free degrees. His construction can be adapted to prove that there exist uncountably many bi-immune free degrees. Hence, $\mathbf{D}-\mathbf{Sp}(\mathcal{F})$ is uncountable. $\square$

Bi-immune but not Hyperimmune

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Question

Is $\mathbf{Sp}(\mathcal{F})$ *exactly the bi-immune degrees?*

A Hyperimmune-like example

Let
$\mathcal{G} = \{\{n\} \oplus F \mid |F| < \infty \wedge (\{D_{\varphi_n(m)}\}_{m\in\omega}$ is a disjoint strong array $\rightarrow (\exists m)[D_{\varphi_n(m)} \subset F])\}$.

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Proposition

There exists an X-computable copy of $\mathcal{G}$ if and only if there exists $Y \equiv_T X$ such that $(\forall e)|Y^{[e]}| < \infty$ and $(\forall e)[\{D_{\varphi_e(m)}\}_{m\in\omega}$ is a disjoint strong array $\rightarrow (\exists m)[D_{\varphi_e(m)} \subset Y^{[e]}]]$.

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Note

We have a computable function $g(i,s)$ such that if φ_i is total then so is $\varphi_{g(i,s)}$ and
$\{\varphi_{g(i,s)}(x)\}_{x\in\omega} = \{\varphi_i(x) | D_{\varphi_i(x)} \cap \{0,\dots,s\} \neq \emptyset\}_{x\in\omega}$.

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That is,

$$\mathbf{Sp}(\mathcal{G}) = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \textit{ is hyperimmune}\}.$$

A hyperhyperimmune-like example

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Lemma

We have a computable function $g(e,s)$ such that if $\{W_{\varphi_e(m)}\}_{m\in\omega}$ is a disjoint weak array, then $\{W_{\varphi_{g(e,s)}(m)}\}_{m\in\omega}$ is also a disjoint weak array, for all $m \in \omega$ $W_{\varphi_{g(e,s)}(m)} \cap \{0,\ldots,s\} = \emptyset$, and for all $l \in \omega$ there exists $m \in \omega$ such that $W_{\varphi_e(m)} \subseteq W_{\varphi_{g(e,s)}(l)}$.

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The bi-immune degrees are a proper subset of the immune degrees. We have a candidate whose degree spectrum *might* be the bi-immune degrees, but we do not know.

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The bi-hyperhyperimmune degrees are a proper subset of the hyperhyperimmune degrees. We have a candidate whose degree spectrum *might* be the bi-hyperhyperimmune degrees, but we don't even know if the bi-hyperhyperimmune degrees are upward closed.

Thank You!