Phase Transitions for Weakly Increasing Sequences

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OUTLINE

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1. Phase Transitions
   - What are phase transitions?
   - Why study phase transitions?
2. Weakly Increasing Sequences
   - The principle
   - Lower bound
   - Upper bound

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   - What are phase transitions?
   - Why study phase transitions?
2. Weakly Increasing Sequences
   - The principle
   - Lower bound
   - Upper bound
3. Related Results and Expectations
   - Sharper threshold region
   - Erdös-Szekeres and Dilworth

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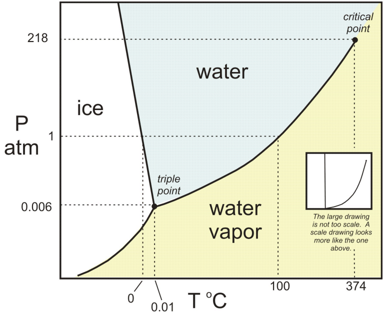
SMALL CHANGES, BIG CONSEQUENCES

- Everyday life: H_2O

P=1atm: ice $\xrightarrow{0^\circ C}$ water $\xrightarrow{100^\circ C}$ water vapor



- Mathematics (statistical physics, evolutionary graph theory, percolation theory, computational complexity, . . .)

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Phase Transitions in Logic and Combinatorics

Parameter f

$$\text{classifiability} \xrightarrow{\text{threshold region for } f} \text{chaos}$$

$$\text{simplicity} \xrightarrow{\text{threshold region for } f} \text{complexity}$$

$$\text{provability} \xrightarrow{\text{threshold region for } f} \text{unprovability}$$

PHASE TRANSITIONS IN LOGIC AND COMBINATORICS

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In this talk

Parameter function f

Statement ISP_f

Theory $\mathrm{I}\Sigma_1$

$$\mathrm{I}\Sigma_1 \vdash \mathrm{ISP}_f \xrightarrow{\textit{threshold region for } f} \mathrm{I}\Sigma_1 \nvdash \mathrm{ISP}_f$$

PHASE TRANSITION FOR ISP_f

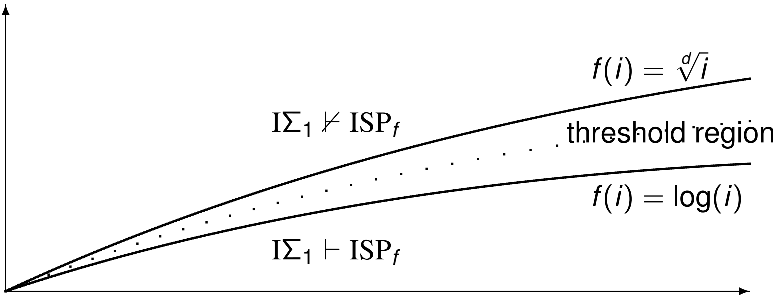

Figure: *Phase transitions for* ISP_f

Phase Transition for ISP_f

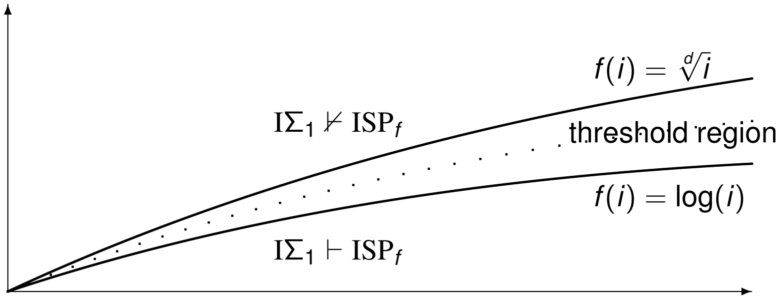

Figure: *Phase transitions for* ISP_f

Goal: Determine the threshold region as exact as possible.

MOTIVATIONS FOR STUDYING PHASE TRANSITIONS

Phase transitions in general

- Phenomenon of universality ($\approx$ in physics):
  Same phase transitions for many theorems from different areas in mathematics.

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Phase transitions in general

- Phenomenon of universality ($\approx$ in physics):
  Same phase transitions for many theorems from different areas in mathematics.
- Understand how to extract complexity from universality, from chaos, from prime numbers, ...

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Phase transition for the principle of weakly increasing sequences

Because of its connection to the open problem on Ramsey's theorem for pairs and two colors (RT^2_2) in reverse mathematics.

MOTIVATIONS FOR STUDYING PHASE TRANSITIONS

Phase transition for the principle of weakly increasing sequences

Because of its connection to the open problem on Ramsey's theorem for pairs and two colors (RT^2_2) in reverse mathematics.

- Simpson [1998]: $n \geq 3$ and $k \geq 2$:

$$\mathrm{RCA}_0 \vdash \mathrm{RT}^n_k \leftrightarrow \mathrm{ACA}_0$$

- Seetapun and Slaman [1995]:

$$\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \nrightarrow \mathrm{ACA}_0$$

Strength RT^2_2?

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Finite Sequences of Natural Numbers

We consider

$$a_0, a_1, a_2, \ldots, a_k$$

with $k \in \mathbb{N}$ and $a_i \in \mathbb{N}$ for $0 \leq i \leq k$.

FINITE SEQUENCES OF NATURAL NUMBERS

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with $k \in \mathbb{N}$ and $a_i \in \mathbb{N}$ for $0 \leq i \leq k$.

Erdös-Szekeres:

$$\text{If } k = n^2 + 1$$
$$\Downarrow$$
$$\exists\, i_0 < i_1 < \ldots < i_n, \text{ such that:}$$
$$a_{i_0} \leq a_{i_1} \leq \ldots \leq a_{i_n}$$

or

$$a_{i_0} > a_{i_1} > \ldots > a_{i_n}.$$

ISP-DENSITY

Definitions

If $f : \mathbb{N} \to \mathbb{N}$ and $X \subseteq \mathbb{N}$, then

- $g : X \to \mathbb{N}$ is called *f*-regressive if

$$g(x) \leq f(x), \text{ for all } x \in X.$$

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- X is called $(n+1)$-ISP-dense(f) if for all f-regressive $g : X \to \mathbb{N}$ there exists a $Y \subseteq X$ such that Y is n-ISP-dense(f) and such that $g \restriction Y$ is weakly increasing.

An Example

Let $f(x) = \sqrt{x}$ and $X = \{5, 6, 8, 11, 35, 108, 167, 201\}$. Is X

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- 1-dense?

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- 1-dense? Yes, because:
  Let $g : X \to \mathbb{N}$ be $\sqrt{\ }$-regressive, i.e.

$$g(x) \leq \sqrt{x}, \text{ for all } x \in X.$$

Now consider the worst case scenario: g is strictly decreasing "as much as possible".

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An Example

Let $f(x) = \sqrt{x}$ and $X = \{5, 6, 7, 11, 35, 108, 167, 201\}$. Is X

- 0-dense? Yes, because:

$$|X| = 9 > \sqrt{5} = \sqrt{\min(X)}.$$

- 1-dense? Yes, because:
  Let $g : X \to \mathbb{N}$ be $\sqrt{\ }$-regressive, i.e.

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Now consider the worst case scenario: g is strictly decreasing "as much as possible".

AN EXAMPLE

$g : \{5, 6, 8, 11, 35, 108, 167, 201\} \rightarrow \mathbb{N}$

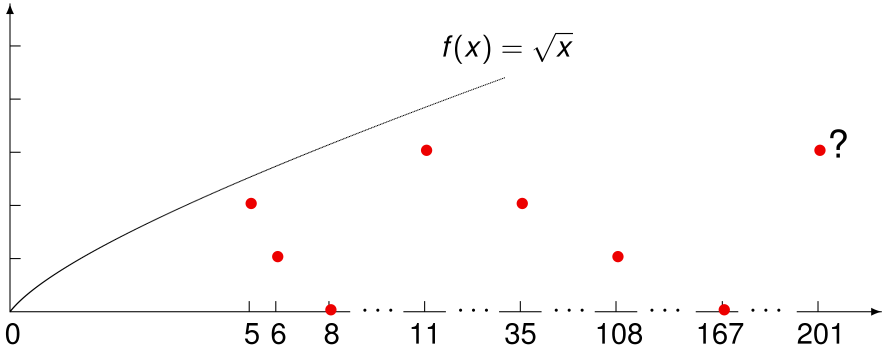

Figure: *g is $\sqrt{}$-regressive*

AN EXAMPLE

Whatever value g may take in $x = 201$, it is always possible to find a Y, such that $|Y| > 2$ and $g \restriction Y$ is weakly increasing.
$\Rightarrow X$ is 1-dense.

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Whatever value g may take in $x = 201$, it is always possible to find a Y, such that $|Y| > 2$ and $g \restriction Y$ is weakly increasing.
$\Rightarrow X$ is 1-dense.

- Is X 2-dense? No, because:
  Constructed g is counterexample

Theorem and Proof

Definition

For any $f : \mathbb{N} \to \mathbb{N}$,

$$\mathrm{ISP}_f := (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-}\mathrm{ISP}\text{-dense}(f)).$$

THEOREM AND PROOF

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Theorem 1

$\mathrm{I}\Sigma_1 \vdash \mathrm{ISP}_{\log}$.

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Theorem 1

$\mathrm{I}\Sigma_1 \vdash \mathrm{ISP}_{\log}$.

Proof

By induction, applying Erdös-Szekeres.

Meaning

Theorem 1 $\Rightarrow$ lower bound

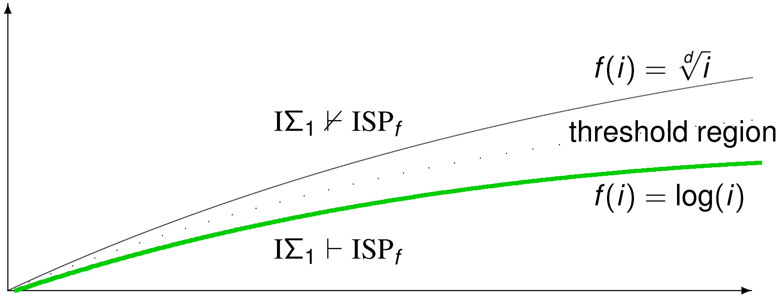

Figure: *Phase transitions for* ISP_f

Theorem and Proof

Theorem 2

Let $d \in \mathbb{N}$. Then

$$\mathrm{I}\Sigma_1 \nvdash \mathrm{ISP}_{\sqrt[d]{\cdot}}.$$

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Theorem 2

Let $d \in \mathbb{N}$. Then

$$\mathrm{I}\Sigma_1 \nvdash \mathrm{ISP}_{\sqrt[d]{\cdot}}.$$

Proof

Lemma 1 $\wedge$ Lemma 2 $\Rightarrow$ Lemma 3.
Lemma 3 $\Rightarrow$ Theorem 2. □

Theorem and Proof

Define

$$\begin{aligned} F_0(i) &:= i+1; \\ F_{k+1}(i) &:= F_k^{\sqrt[d]{i}}(i); \\ F(i) &:= F_i(i). \end{aligned}$$

Theorem and Proof

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\end{aligned}
$$

Then

$$
\begin{aligned}
F_i(n) &\approx A_i(n) \\
F(n) &\approx Ack(n)
\end{aligned}
$$

$A_i = i$th approximation of the Ackermann function Ack.

Theorem and Proof

Lemma 1 (Informal)

$[a, b]$ n-ISP-dense($\sqrt[d]{\ }$)
$\Rightarrow \exists Y \subseteq [a, b]$: Y has nice properties.

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Lemma 1 (Formal)

$[a, b]$ n-ISP-dense($\sqrt[d]{\ }$)
$\Rightarrow \exists Y \subseteq [a, b]$: Y is $(n-1)$-ISP-dense($\sqrt[d]{\ }$) and

$$\forall i(F_1^{i+1}(a) \leq b \rightarrow |Y \cap [F_1^i(a), F_1^{i+1}(a)[| = 1)$$

Theorem and Proof

Proof

Define $G : [a, b] \mapsto \mathbb{N}$, such that G is $\sqrt[d]{}$-regressive, and:

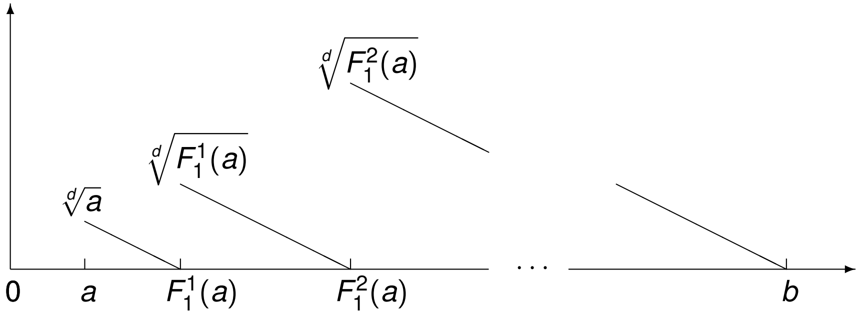

Figure: *Proof of Lemma 1*

Theorem and Proof

Lemma 2

$\approx$ Lemma 1, but we start from an $(n-k)$-ISP-dense$(\sqrt[d]{\ })$ set, $0 < k \leq n$.

Proof

$\approx$ Lemma 1, but we need F_k's, $0 < k \leq n$. □

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Proof

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Lemma 3

$(Y \subseteq [a, b]$ is n-ISP-dense($\sqrt[d]{\ }$) $\wedge\ a \geq 1) \Rightarrow \max(Y) \geq F_{n+1}(a)$.

Proof

Combine Lemma 1 and Lemma 2. □

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THEOREM AND PROOF

Definition

$\mathrm{ISP}_f(n, a) :=$ the least natural number b, such that $[a, b]$ is n-ISP-dense(f).

Let $a, n \in \mathbb{N}$:

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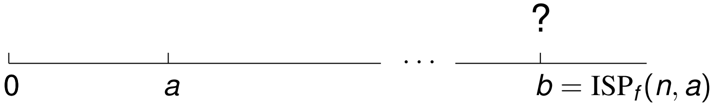

Remark

$$(\mathrm{I}\Sigma_1 \vdash \mathrm{ISP}_f) \Leftrightarrow (\mathrm{ISP}_f(n, n) \text{ is primitive recursive.})$$

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Proof of Theorem 2

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&\Rightarrow \quad \mathrm{ISP}_{\sqrt[d]{}} \text{ is not primitive recursive}
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□

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MEANING

Theorem 2 $\Rightarrow$ upper bound

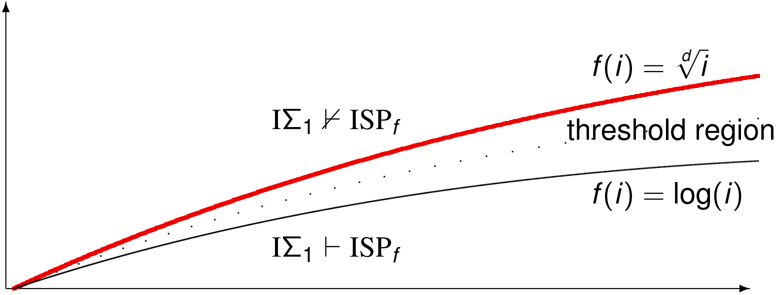

Figure: *Phase transitions for* ISP_f

Sharpening the Threshold Region

Claim 1

Let d be a natural number and $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$. Then

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Claim 2

Let $f(i) = i^{\frac{1}{Ack^{-1}(i)}}$. Then

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Sharpening the Threshold Region

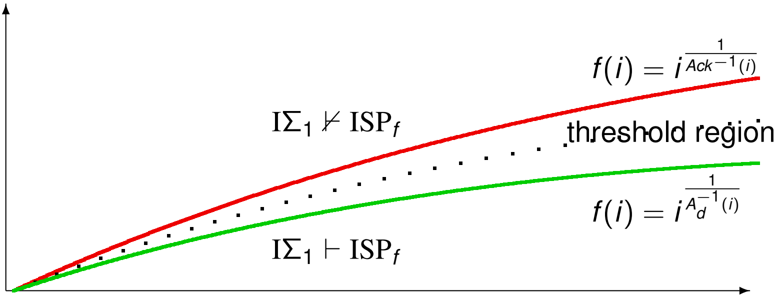

Figure: *Phase transitions for* ISP_f

Related Results

Instead of $F : X \rightarrow \mathbb{N}$, we now consider $F : X \rightarrow \omega^l$, where ω is the first infinite ordinal and $l \in \mathbb{N}$.

RELATED RESULTS

Instead of $F : X \to \mathbb{N}$, we now consider $F : X \to \omega^l$, where ω is the first infinite ordinal and $l \in \mathbb{N}$.
$\Rightarrow$ definition of ω^l-n-ISP-density-(f).

Similar results can be obtained.

Erdös-Szekeres

The Erdös-Szekeres theorem states that a given sequence $a_0, \ldots, a_{n^2}$ of real numbers contains a weakly increasing subsequence of length $n+1$ or a strictly decreasing subsequence of length $n+1$.

$\Rightarrow$ ES-density

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$\Rightarrow$ ES-density

Dilworth

The Dilworth theorem states that a given partial order with distinct elements $a_0, \ldots, a_{n^2}$ contains a chain of length $n+1$ or an antichain of length $n+1$.

$\Rightarrow$ D-density

Thanks - Personal Information

Thank you for listening.

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