Ramsey's theorem and cone avoidance

Damir Dzhafarov

Department of Mathematics
University of Chicago

8 July, 2008/Logic Colloquium, Bern

Background

Definition

Let $X \subseteq \omega$ be an infinite set and $n, k \in \omega$.

1. $[X]^n := \{Y \subset X : |Y| = n\}$.
2. A *k-coloring on X of exponent n* is a function $f : [X]^n \to k = \{0, \ldots, k-1\}$.
3. A set $H \subseteq X$ is *homogeneous for f* if $f \upharpoonright [H]^n$ is constant.

Background

Definition

Let $X \subseteq \omega$ be an infinite set and $n, k \in \omega$.

1. $[X]^n := \{Y \subset X : |Y| = n\}$.
2. A *k-coloring on X of exponent n* is a function $f : [X]^n \to k = \{0, \ldots, k-1\}$.
3. A set $H \subseteq X$ is *homogeneous for f* if $f \upharpoonright [H]^n$ is constant.

Ramsey's theorem

For every $n, k \geq 1$, every $f : [\omega]^n \to k$ admits an infinite homogeneous set.

Background

- We restrict to computable colorings and ask about the possible complexity of infinite homogeneous sets.

Background

- We restrict to computable colorings and ask about the possible complexity of infinite homogeneous sets.

Theorem (Specker, 1969)

There exists a computable 2-coloring of $[\omega]^2$ admitting no infinite computable homogeneous set.

Background

- We restrict to computable colorings and ask about the possible complexity of infinite homogeneous sets.

Theorem (Specker, 1969)

There exists a computable 2-coloring of $[\omega]^2$ admitting no infinite computable homogeneous set.

Theorem (Jockusch, 1972)

There exists a computable 2-coloring of $[\omega]^2$ admitting no infinite homogeneous set computable in $0'$.

Cone non-avoidance

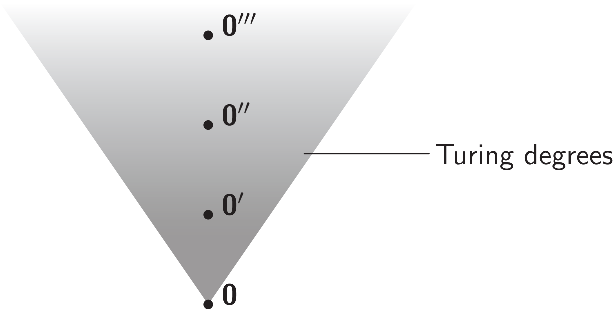
Theorem (Jockusch, 1972)

For every $n \geq 3$, there exists a computable 2-coloring of $[\omega]^n$ all of whose infinite homogeneous sets compute $0^{(n-2)}$.

Cone non-avoidance

Theorem (Jockusch, 1972)

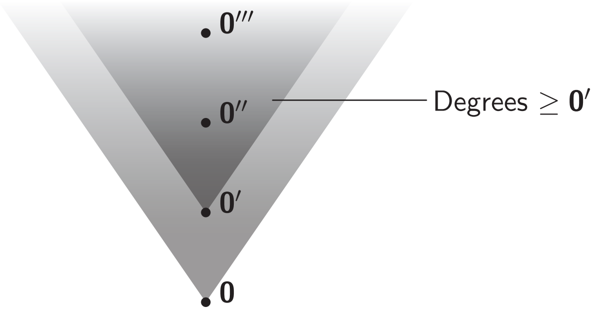
For every $n \geq 3$, there exists a computable 2-coloring of $[\omega]^n$ all of whose infinite homogeneous sets compute $0^{(n-2)}$.



Cone non-avoidance

Theorem (Jockusch, 1972)

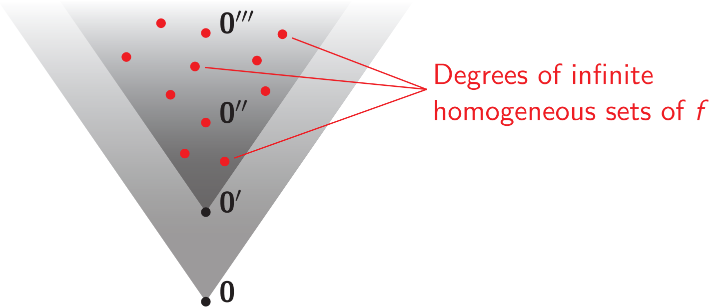
For every $n \geq 3$, there exists a computable 2-coloring of $[\omega]^n$ all of whose infinite homogeneous sets compute $0^{(n-2)}$.



Cone non-avoidance

Theorem (Jockusch, 1972)

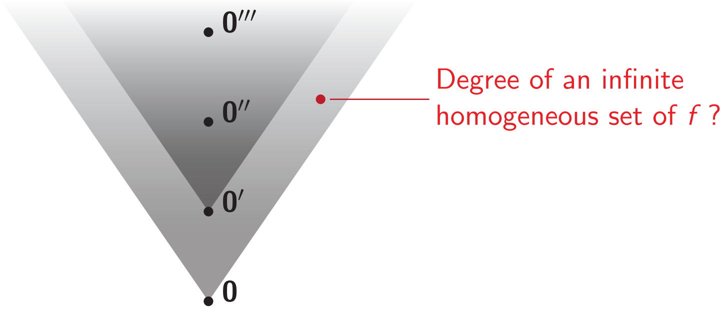
For every $n \geq 3$, there exists a computable 2-coloring of $[\omega]^n$ all of whose infinite homogeneous sets compute $0^{(n-2)}$.



Cone avoidance

Question (Jockusch, 1972)

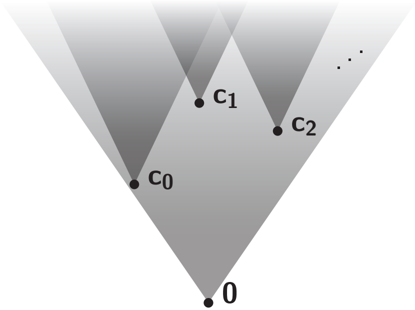
Does every computable 2-coloring of $[\omega]^2$ *admit an infinite homogeneous set which does not compute* $0'$*?*



Cone avoidance

Theorem (Seetapun, 1995)

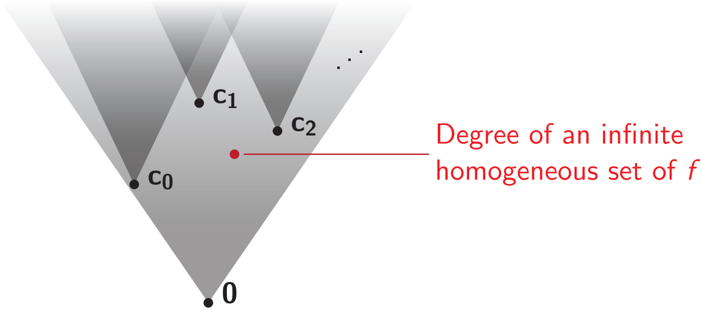
Given $C_0, C_1, \ldots >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $C_i \not\leq_T H$ for all i.



Cone avoidance

Theorem (Seetapun, 1995)

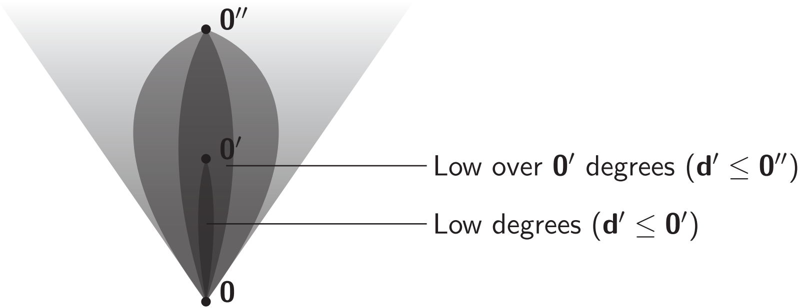
Given $C_0, C_1, \ldots >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $C_i \not\leq_T H$ for all i.



Lowness over $0^{(n)}$

Theorem (Jockusch, 1972)

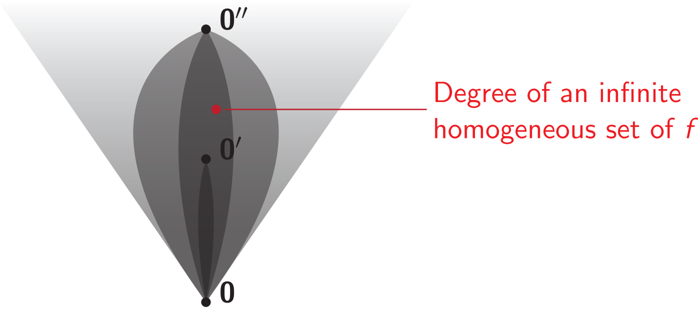
For every $n, k \geq 2$, every computable $f : [\omega]^n \to k$ has an infinite homogeneous set H with $H' \leq_T 0^{(n)}$.



Lowness over $0^{(n)}$

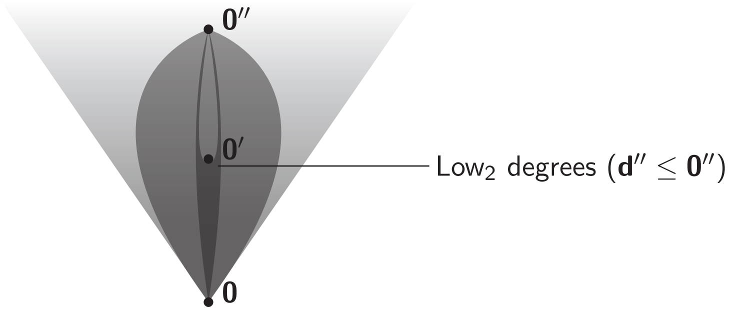
Theorem (Jockusch, 1972)

For every $n, k \geq 2$, every computable $f : [\omega]^n \to k$ has an infinite homogeneous set H with $H' \leq_T 0^{(n)}$.



Question (Jockusch, 1972)

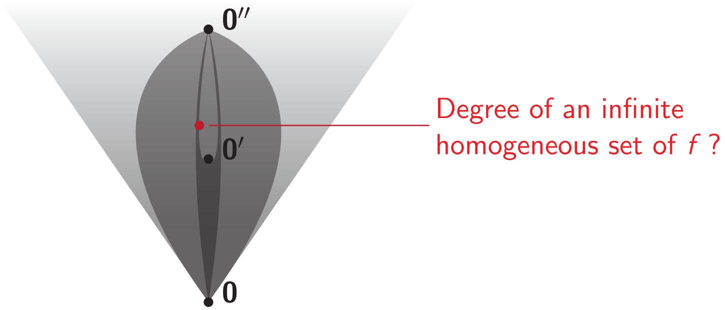
Does every computable 2-coloring of $[\omega]^2$ admit an infinite homogeneous set H with $H'' \leq_T 0''$ (i.e., which is low$_2$)?



Low$_2$ness

Question (Jockusch, 1972)

Does every computable 2-coloring of $[\omega]^2$ admit an infinite homogeneous set H with $H'' \leq_T 0''$ (i.e., which is low$_2$)?



Low$_2$ness

Recall...

Definition

A degree $\mathbf{d}$ is *PA over* $\mathbf{0}'$, written $\mathbf{d} \gg \mathbf{0}'$, if every infinite $0'$-computable tree in $2^{<\omega}$ has an infinite path of degree $\leq \mathbf{d}$.

Recall...

Definition

A degree $\mathbf{d}$ is *PA over* $\mathbf{0}'$, written $\mathbf{d} \gg \mathbf{0}'$, if every infinite $0'$-computable tree in $2^{<\omega}$ has an infinite path of degree $\leq \mathbf{d}$.

- There exists an infinite $0'$-computable tree in $2^{<\omega}$ each of whose infinite paths has degree $\gg \mathbf{0}'$.

Low$_2$ness

Recall...

Definition

A degree $\mathbf{d}$ is *PA over* $\mathbf{0}'$, written $\mathbf{d} \gg \mathbf{0}'$, if every infinite $0'$-computable tree in $2^{<\omega}$ has an infinite path of degree $\leq \mathbf{d}$.

- There exists an infinite $0'$-computable tree in $2^{<\omega}$ each of whose infinite paths has degree $\gg \mathbf{0}'$.
- By the Low Basis Theorem relativized to $\mathbf{0}'$, there exists a degree $\mathbf{d} \gg \mathbf{0}'$ which is low over $\mathbf{0}'$ (i.e., $\mathbf{d}' \leq \mathbf{0}''$).

Low$_2$ness

Recall...

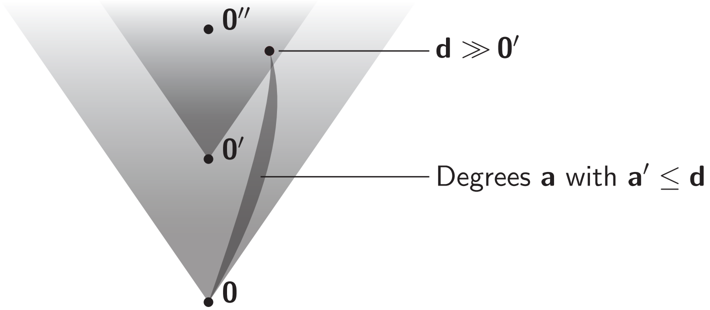
Definition

A degree $\mathbf{d}$ is *PA over* $\mathbf{0}'$, written $\mathbf{d} \gg \mathbf{0}'$, if every infinite $0'$-computable tree in $2^{<\omega}$ has an infinite path of degree $\leq \mathbf{d}$.

- There exists an infinite $0'$-computable tree in $2^{<\omega}$ each of whose infinite paths has degree $\gg \mathbf{0}'$.
- By the Low Basis Theorem relativized to $\mathbf{0}'$, there exists a degree $\mathbf{d} \gg \mathbf{0}'$ which is low over $\mathbf{0}'$ (i.e., $\mathbf{d}' \leq \mathbf{0}''$).
- If $\mathbf{a}$ is low over $\mathbf{0}'$ and $\mathbf{b}' \leq \mathbf{a}$ then $\mathbf{b}'' \leq \mathbf{a}' \leq \mathbf{0}''$, so $\mathbf{b}$ is low$_2$.

Theorem (Cholak, Jockusch, and Slaman, 2001)

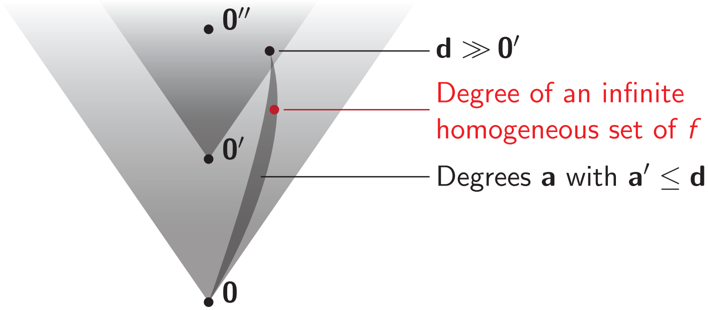
Given $\mathbf{d} \gg \mathbf{0}'$, *every computable* $f : [\omega]^2 \to 2$ *admits an infinite homogeneous set* H *with* $\deg(H)' \leq \mathbf{d}$.



Low$_2$ness

Theorem (Cholak, Jockusch, and Slaman, 2001)

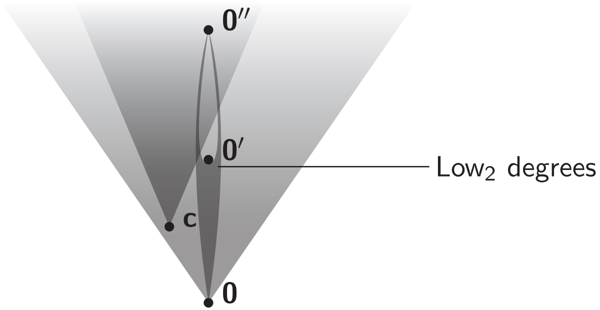
Given $\mathbf{d} \gg \mathbf{0}'$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$.



Cone avoidance and low$_2$ness

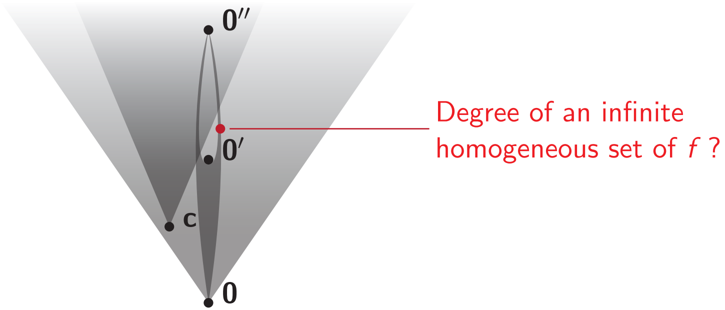
Question (Cholak, Jockusch, and Slaman, 2001)

Given $C >_T 0$, does every computable $f : [\omega]^2 \to 2$ admit an infinite low$_2$ homogeneous set H with $C \not\leq_T H$?



Question (Cholak, Jockusch, and Slaman, 2001)

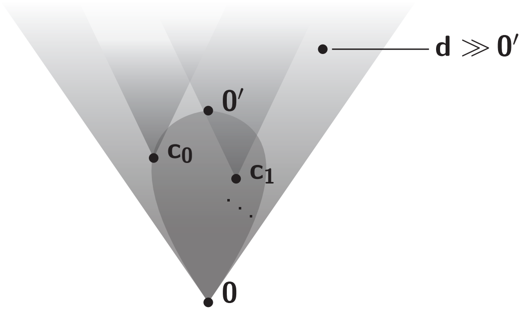
Given $C >_T 0$, does every computable $f : [\omega]^2 \to 2$ admit an infinite low$_2$ homogeneous set H with $C \not\leq_T H$?



Cone avoidance and low$_2$ness

Theorem (Dzhafarov and Jockusch)

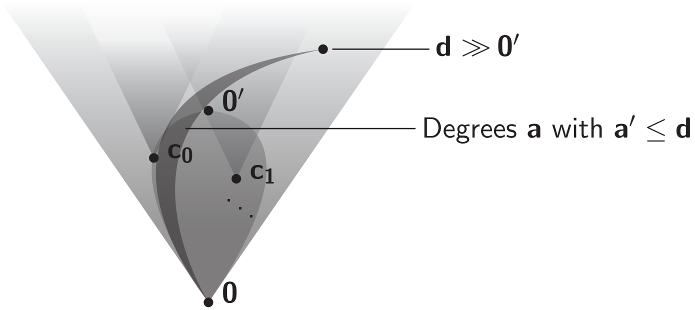
Given $C_0, C_1, \ldots >_T 0$ with $\oplus_i C_i \leq_T 0'$ and $\mathbf{d} \gg \mathbf{0}'$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C_i \not\leq_T H$ for all i.



Cone avoidance and low$_2$ness

Theorem (Dzhafarov and Jockusch)

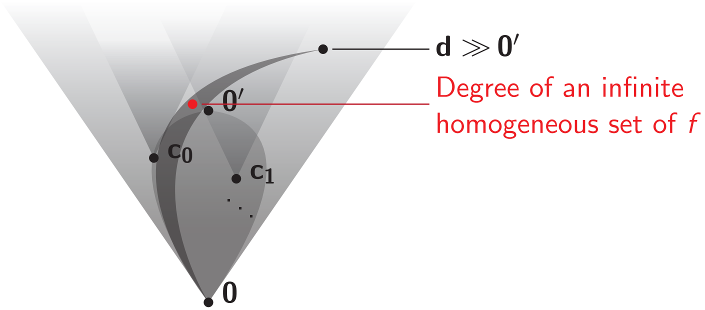
Given $C_0, C_1, \ldots >_T 0$ with $\oplus_i C_i \leq_T 0'$ and $\mathbf{d} \gg \mathbf{0}'$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C_i \not\leq_T H$ for all i.



Cone avoidance and low$_2$ness

Theorem (Dzhafarov and Jockusch)

Given $C_0, C_1, \ldots >_T 0$ with $\oplus_i C_i \leq_T 0'$ and $\mathbf{d} \gg \mathbf{0}'$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C_i \not\leq_T H$ for all i.



Cone avoidance and low$_2$ness

Corollary (Dzhafarov and Jockusch)

Given $0 <_T C \leq_T 0'$, every computable $f : [\omega]^2 \to 2$ admits an infinite low$_2$ homogeneous set H with $C \not\leq_T H$.

Cone avoidance and low$_2$ness

Corollary (Dzhafarov and Jockusch)

Given $0 <_T C \leq_T 0'$, every computable $f : [\omega]^2 \to 2$ admits an infinite low$_2$ homogeneous set H with $C \not\leq_T H$.

- The case $C \not\leq_T 0'$ is handled by analyzing the subcases $C \leq_T 0''$ and $C \not\leq_T 0''$.

Theorem (Dzhafarov and Jockusch)

Given $C >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite low$_2$ homogeneous set H with $C \not\leq_T H$.

- A more careful case analysis yields the following extension:

Theorem (Dzhafarov and Jockusch)

Given $C_0, \ldots, C_n >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite low$_2$ homogeneous set H with $C_i \not\leq_T H$ for each $i \leq n$.

Recall...

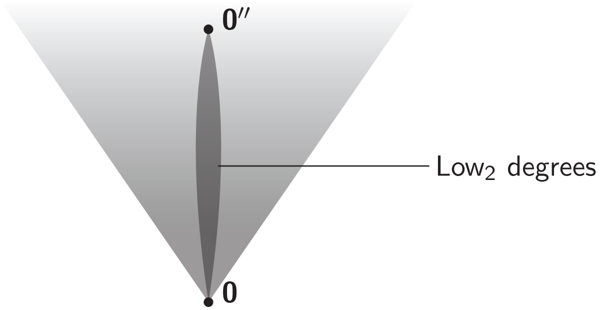
Definition

Degrees $\mathbf{a}$ and $\mathbf{b}$ form *a minimal pair* if $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.

Low$_2$ness and minimal pairs

Theorem (Dzhafarov and Jockusch)

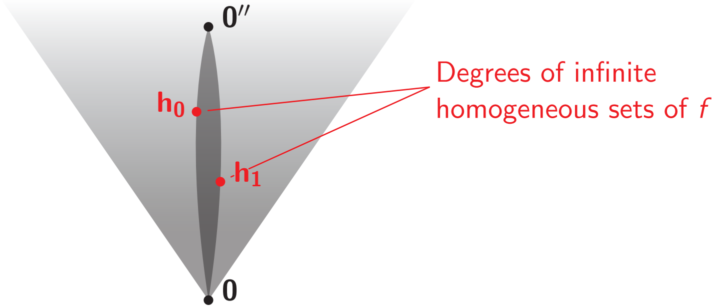
Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



Low$_2$ness and minimal pairs

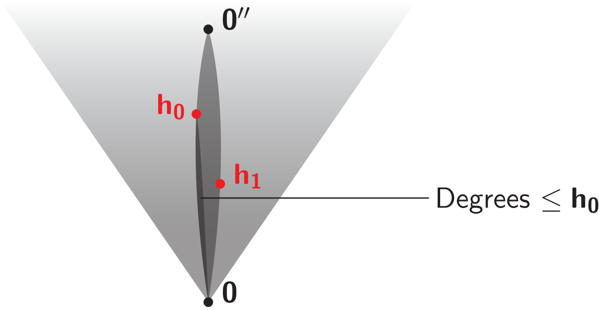
Theorem (Dzhafarov and Jockusch)

Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



Theorem (Dzhafarov and Jockusch)

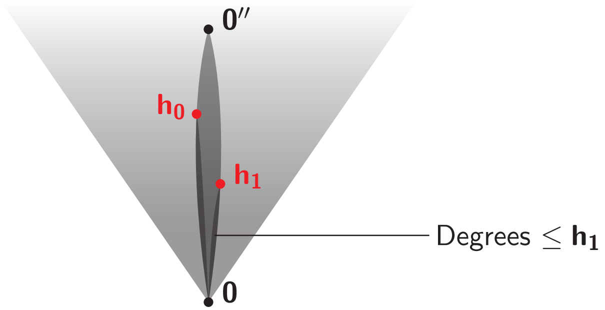
Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



Low$_2$ness and minimal pairs

Theorem (Dzhafarov and Jockusch)

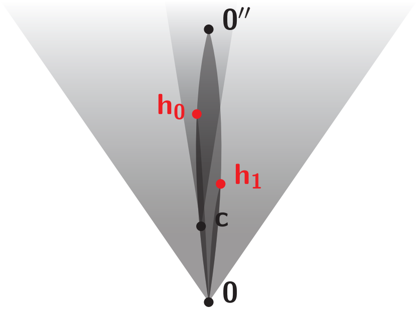
Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



Low$_2$ness and minimal pairs

Theorem (Dzhafarov and Jockusch)

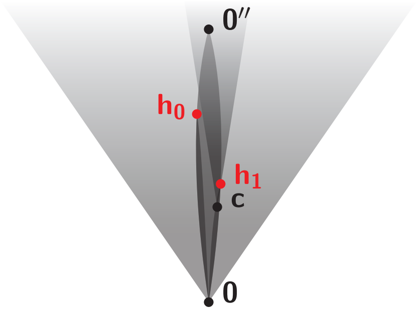
Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



Low$_2$ness and minimal pairs

Theorem (Dzhafarov and Jockusch)

Every computable $f : [\omega]^2 \to 2$ *admits infinite low*$_2$ *homogeneous sets* H_0, H_1 *such that* $\deg(H_0)$ *and* $\deg(H_1)$ *form a minimal pair.*



Questions

Open question

Given any noncomputable set C and any degree $\mathbf{d} \gg \mathbf{0}'$, does every computable $f : [\omega]^2 \to 2$ admit an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C \not\leq_T H$?

Questions

Open question

Given any noncomputable set C and any degree $\mathbf{d} \gg \mathbf{0}'$, does every computable $f : [\omega]^2 \to 2$ admit an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C \not\leq_T H$?

Open question (Simpson)

Does every computable $f : [\omega]^2 \to 2$ admit infinite homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair and $H_0 \oplus H_1$ is low$_2$?

References

P. A. Cholak, C. G. Jockusch, Jr., and T. A. Slaman
On the strength of Ramsey's theorem for pairs.
J. Symbolic Logic **66** (2001), no. 1, 1–55

D. D. Dzhafarov and C. G. Jockusch, Jr.
Ramsey's theorem and cone avoidance
Submitted

C. G. Jockusch, Jr.
Ramsey's theorem and recursion theory.
J. Symbolic Logic **37** (1972), no. 2, 268–280

Thank you for your attention.