The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Alessandro Facchini$^1$ Luca Alberucci$^2$

$^1$ University of Lausanne and LaBRI, Bordeaux
$^2$ IAM, University of Berne

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Goal of our works
Understand the expressive power of modal $\mu$-calculus over different classes of models.
What is the modal $\mu$-calculus
What is the modal $\mu$-calculus

the propositional modal $\mu$-calculus

$=$

propositional modal logic

$+$

least and greatest fixpoint operators
Expressive power

Eventually "p":

$$\mu x. p \lor \diamond x$$

Allways "p":

$$\nu x. p \land \Box x$$

Allways eventually "p":

$$\nu x. (\mu y. p \lor \diamond y) \land \Box x$$

There is a branch such that infinitely often "p":

$$\nu x. \mu y. (p \land \diamond x) \lor \diamond y$$
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Introduction

The fixpoint alternation depth

The fixpoint alternation of a formula is the number of non-trivial nestings of alternating least and greatest fixpoints.
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Introduction

Example

$\varphi_1 := p \lor \Diamond q$
Example

\[ \varphi_2 \equiv \mu x. p \lor \Diamond x \]
Example

\[ \varphi_3 := \nu x. \mu y. (p \land \Diamond x) \lor \Diamond y \]
Example

\[ \varphi_4 := \mu x (\nu y (p \land \Diamond y) \lor \Box x) \]
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Introduction
Is the **semantical** hierarchy strict?
Semantical complexity

Bradfield (1996): Strictness of semantical modal $\mu$-calculus hierarchy

The semantical modal $\mu$-calculus hierarchy is strict on the class of all transition systems.

$\Rightarrow$ For each $n$ there is a formula $\varphi$ with $\text{ad}(\varphi) = n$ such that for all formulae $\psi$ with $\text{ad}(\psi) < n$ we do not have

$$\text{for all transition systems } T : \quad (T \models \varphi \iff T \models \psi).$$

Proof.

Game formulae are complete for their corresponding level. $\square$
### Question

What happens for restricted classes of transition systems?

<table>
<thead>
<tr>
<th></th>
<th>refl</th>
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<th>sym &amp; tr</th>
<th>tr &amp; wf</th>
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</table>
Overview

The modal $\mu$-calculus

The Hierarchy on Transitive and Symmetric Transition Systems

The Hierarchy on Transitive Transition Systems

A final picture
Syntax of $\mu$-calculus

$\varphi ::= p \mid \neg p \mid \top \mid \bot \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid \Diamond \varphi \mid \Box \varphi \ldots$

$\ldots \mid \mu x. \varphi \mid \nu x. \varphi$

where $p, x \in \text{Prop}$ and $x$ occurs only positively.
A transition system $\mathcal{T}$ is a triple $(S, \rightarrow^{\mathcal{T}}, \lambda^{\mathcal{T}})$ consisting of

- a set $S$ of states,
- a binary relation $\rightarrow^{\mathcal{T}} \subseteq S \times S$ called transition relation,
- the valuation $\lambda : P \rightarrow \wp(S)$ assigning to each propositional variable $p$ a subset $\lambda(p)$ of $S$. 

Semantics
Denotation and validity of a formula

Given a transition system $\mathcal{T} = (S, \rightarrow^{\mathcal{T}}, \lambda^{\mathcal{T}})$. We define $\| \varphi \|_{\mathcal{T}}$ inductively such that

- $\| p \|_{\mathcal{T}} = \lambda^{\mathcal{T}}(p)$ and $\| \sim p \|_{\mathcal{T}} = S \setminus \lambda^{\mathcal{T}}(p)$
- $\| \alpha \land \beta \|_{\mathcal{T}} = \| \alpha \|_{\mathcal{T}} \cap \| \beta \|_{\mathcal{T}}$ and $\| \alpha \lor \beta \|_{\mathcal{T}} = \| \alpha \|_{\mathcal{T}} \cup \| \beta \|_{\mathcal{T}}$
- $\| \Box \alpha \|_{\mathcal{T}} = \{ s \in S \mid \forall s'(s \rightarrow^{\mathcal{T}} s' \implies s' \in \| \alpha \|_{\mathcal{T}}) \}$
- $\| \Diamond \alpha \|_{\mathcal{T}} = \{ s \in S \mid \exists s'(s \rightarrow^{\mathcal{T}} s' \text{ and } s' \in \| \alpha \|_{\mathcal{T}}) \}$. 
Denotation and validity of a formula

\[ \| \nu x. \varphi(x) \|_T = \begin{cases} \bigcup \{ S' \subseteq S \mid S' \subseteq \| \varphi(x) \|_T[x \mapsto S'] \} \\ GFP(\| \varphi(x) \|_T) \end{cases} \]

\[ \| \mu x. \varphi(x) \|_T = \begin{cases} \bigcap \{ S' \subseteq S \mid \| \varphi(x) \|_T[x \mapsto S'] \subseteq S' \} \\ LFP(\| \varphi(x) \|_T) \end{cases} \]

...
An equivalence

A formula is **well-named** if bound and free variables are pairwise distinct and if all bound variable occur only once and are guarded.

**Lemma**

*Every formula $\varphi$ is equivalent to a well-named formula $nf(\varphi)$.***
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The modal $\mu$-calculus

Evaluation games

- The player are $V$ (verifier) and $F$ (falsifier). Consider $w \in S$, $V$ tries to show that $w \vDash \varphi$, while $F$ tries to show that $w \not\vDash \varphi$. 
Evaluation games

The player are $\mathbf{V}$ (verifier) and $\mathbf{F}$ (falsifier). Consider $w \in S$, $\mathbf{V}$ tries to show that $w \models \varphi$, while $\mathbf{F}$ tries to show that $w \not\models \varphi$.

- the play starts at $\langle \varphi, w_0 \rangle$
- the admissible moves are choices of subformulas and points of $S$ which respect the following rules:
Evaluation game for modal logic

<table>
<thead>
<tr>
<th>position</th>
<th>player</th>
<th>next position</th>
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</thead>
<tbody>
<tr>
<td>$\langle p_i, w \rangle$</td>
<td>$\mathbf{F}$ iff $p_i \in \lambda(w)$</td>
<td>-</td>
</tr>
<tr>
<td>$\langle \neg p_i, w \rangle$</td>
<td>$\mathbf{V}$ iff $p_i \in \lambda(w)$</td>
<td>-</td>
</tr>
<tr>
<td>$\langle \psi \lor \phi, w \rangle$</td>
<td>$\mathbf{V}$ chooses between $\langle \psi, w \rangle$ and $\langle \phi, w \rangle$</td>
<td>$\mathbf{V}$ choice</td>
</tr>
<tr>
<td>$\langle \psi \land \phi, w \rangle$</td>
<td>$\mathbf{F}$ chooses between $\langle \psi, w \rangle$ and $\langle \phi, w \rangle$</td>
<td>$\mathbf{F}$ choice</td>
</tr>
<tr>
<td>$\langle \lozenge \psi, w \rangle$</td>
<td>$\mathbf{V}$ chooses a point $w'$ s.t. $wRw'$</td>
<td>$\langle \psi, w' \rangle$</td>
</tr>
<tr>
<td>$\langle \Box \psi, w \rangle$</td>
<td>$\mathbf{F}$ chooses a point $w'$ s.t. $wRw'$</td>
<td>$\langle \psi, w' \rangle$</td>
</tr>
</tbody>
</table>
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The modal $\mu$-calculus

Evaluation game for modal logic

- $V$ wins iff $F$ cannot move.
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Example

$(\Diamond \Diamond p \lor \Box p) \land r$
### Evaluation game for μ-calculus

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>⟨pi, w⟩</td>
<td>F iff pi ∈ λ(w)</td>
<td>-</td>
</tr>
<tr>
<td>⟨¬pi, w⟩</td>
<td>V iff pi ∈ λ(w)</td>
<td>-</td>
</tr>
<tr>
<td>⟨ψ ∨ φ, w⟩</td>
<td>V chooses between ⟨ψ, w⟩ and ⟨φ, w⟩</td>
<td>V choice</td>
</tr>
<tr>
<td>⟨ψ ∧ φ, w⟩</td>
<td>F chooses between ⟨ψ, w⟩ and ⟨φ, w⟩</td>
<td>F choice</td>
</tr>
<tr>
<td>⟨◊ψ, w⟩</td>
<td>V chooses a point w' s.t. wRw'</td>
<td>⟨ψ, w'⟩</td>
</tr>
<tr>
<td>⟨□ψ, w⟩</td>
<td>F chooses a point w' s.t. wRw'</td>
<td>⟨ψ, w'⟩</td>
</tr>
<tr>
<td>⟨μx.ψ, w⟩</td>
<td>-</td>
<td>⟨ψ, w⟩</td>
</tr>
<tr>
<td>⟨νx.ψ, w⟩</td>
<td>-</td>
<td>⟨ψ, w⟩</td>
</tr>
<tr>
<td>⟨x, w⟩</td>
<td>-</td>
<td>⟨ηx.ψ(x), w⟩</td>
</tr>
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</table>
Evaluation game for $\mu$-calculus

- the game behind the $\mu$-calculus is a **parity game**
- “$\mu$ means **finite** looping”
- “$\nu$ means **infinite** looping”
$\Omega : V \rightarrow \omega$ is priority function defined on $\langle \eta x. \delta, s \rangle \in V$, where $\eta \in \{\mu, \nu\}$. In this case we have that:

$$\Omega(\langle \eta x. \delta, s \rangle) = \begin{cases} 
\text{ad}(\eta x. \delta) & \text{if } \eta = \mu \text{ and ad is odd, or } \\
\eta = \nu \text{ and ad is even;} \\
\text{ad}(\eta x. \delta) - 1 & \text{if } \eta = \mu \text{ and ad is even, or } \\
\eta = \nu \text{ and ad is odd.}
\end{cases}$$
Winning Conditions

Given a play $\pi$ of $E(\varphi, (T, w_I))$

1. if $\pi$ is finite, $V$ wins iff $F$ can’t move anymore;
2. if $\pi$ is infinite iff $V$ wins iff the highest priority appearing infinitely often is even.
Example

\[ \mu x (p \lor \Box x) \]
Example

$$\mu x (p \lor \Diamond x)$$
Game-theoretical version of the “fundamental theorem”

Theorem [Streett Emerson 89]

\( s \in ||\varphi||_T \) iff \( V \) has a winning strategy in \( E(\varphi, (T, w_I)) \).
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on Transitive and Symmetric Transition Systems
Theorem (AF (08\textsuperscript{a})))

Let $\mathcal{T}$ be a transitive and symmetric transition system. We have that

$$\llbracket \nu x. \varphi(x) \rrbracket_{\mathcal{T}} = \llbracket \varphi(\varphi(\top)) \rrbracket_{\mathcal{T}}.$$
The syntactical translation \((\cdot)^t : \mathcal{L}_\mu \to \mathcal{L}_M\) is defined as:

\[\begin{align*}
\vdash \ldots \\
\vdash (\mu x. \varphi)^t &= (\varphi(\varphi(\bot)))^t \\
\vdash (\nu x. \varphi)^t &= (\varphi(\varphi(\top)))^t
\end{align*}\]
A first collapse

Corollary (AF (08a))

On transitive and symmetric (and reflexive) transition systems we have that

$$\|\varphi\|_I = \|\varphi^t\|_I.$$
The Hierarchy on Transitive Transition Systems
What was already known

Theorem (Lenzi (96), Lenzi-d’Agostino (08))

For every class $A$ of transitive transition systems, $A$ is definable by a $\mu$-formula iff it is recognizable by Büchi automata.
Conjecture

Over transitive transition systems, the $\mu$-calculus collapses to its alternation free fragment.
Lemma
Let $\mathcal{T}$ be a transitive transition system and let $s, s'$ be two states such that $s \xrightarrow{\mathcal{T}} s'$. For all $\mu$-formulae $\varphi$ we have that

$$s \in \|\Box \varphi\|_{\mathcal{T}} \implies s' \in \|\Box \varphi\|_{\mathcal{T}}$$

and

$$s' \in \|\Diamond \varphi\|_{\mathcal{T}} \implies s \in \|\Diamond \varphi\|_{\mathcal{T}}.$$

Theorem (AF $^{(08^a)}$)
Let $\mathcal{T}$ be a transitive transition system and let $\nu x. \varphi(x)$ be a formula such that $x$ is in the scope of a $\Box$ modality. We have that

$$\|\nu x. \varphi(x)\|_{\mathcal{T}} = \|\varphi(\varphi(\top))\|_{\mathcal{T}}.$$
$\tau : \mathcal{L}_\mu \rightarrow \mathcal{L}_\mu$ is defined as:

- $\ldots$
- $\tau(\mu x.\varphi) = \tau(\varphi(\varphi(\bot)))$, $x$ is in the scope of a $\Diamond$ in $\varphi$
- $\tau(\mu x.\varphi) = \mu x.\tau(\varphi)$, $x$ is not in the scope of a $\Diamond$ in $\varphi$
- $\tau(\nu x.\varphi) = \tau(\varphi(\varphi(\top)))$, $x$ is in the scope of a $\Box$ in $\varphi$
- $\tau(\nu x.\varphi) = \nu x.\tau(\varphi)$, $x$ is not in the scope of a $\Box$ in $\varphi$
A first step towards the collapse

Corollary
On transitive transition systems we have that

$$\| \varphi \|_I = \| \tau(\varphi) \|_I.$$
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on Transitive Transition Systems

Finite model theorem

Theorem (AF (08a))

For all modal $\mu$-formulae $\varphi$ for which there is a transitive transition system $\mathcal{T}$ and a state $s$ in $\mathcal{T}$ such that $s \in \|\varphi\|_\mathcal{T}$ there is a finite transitive transition system $\mathcal{T}^F$ and a state $s^F$ such that $s^F \in \|\varphi\|_{\mathcal{T}^F}$.

Proof.
For every $\varphi$, $s \in \|\varphi^{tr}\|_\mathcal{T}$ iff $s \in \|\varphi\|_{(\mathcal{T})^{tr}}$. Then, use the fmp of the $\mu$-calculus. \qed
Normalizing winning strategies

Let $\mathcal{T}$ be a finite transitive transition system and $\varphi$ a $\mu$-formula. Consider an arbitrary (memoryless) strategy $\sigma$ for Player 0, not necessarily winning. We define the restriction of $E(\varphi, (\mathcal{T}, s_0))$ on $\sigma$, denoted by $E|_\sigma(\varphi, (\mathcal{T}, s_0))$, the parity game starting from $\langle \varphi, s_0 \rangle$ induced by $\sigma$. 
Normalizing winning strategies

Given a winning strategy $\sigma$ for player 0, we define a measure $d$ on $E|_\sigma(\varphi, (T, s_0))$: 
Normalizing winning strategies

1. $\text{scc}(\langle \psi, s \rangle) = \emptyset$ :
   
   $$d(\langle \psi, s \rangle) = \begin{cases} 
   0 & \text{if } E|_\sigma(\langle \psi, s \rangle) = \emptyset \\
   \max\{d(\langle \phi, s' \rangle) : \langle \phi, s' \rangle \in E|_\sigma(\langle \psi, s \rangle)\} + 1 & \text{else} 
   \end{cases}$$

2. $\text{scc}(\psi, s) \neq \emptyset$ :

   $$d(\langle \psi, s \rangle) = 0 \quad \text{if } \bigcup\{E|_\sigma(\langle \alpha, s \rangle) : \langle \alpha, s \rangle \in \text{scc}(\psi, s)\} \setminus \text{scc}(\psi, s) = \emptyset,$$

   else

   $$d(\langle \psi, s \rangle) = \max\{d(\langle \phi, s' \rangle) : \langle \phi, s' \rangle \notin \text{scc}(\langle \psi, s \rangle) \text{ and exists } \langle \xi, s'' \rangle \in \text{scc}(\langle \psi, s \rangle) \text{ with } \langle \phi, s' \rangle \in E|_\sigma(\langle \xi, s'' \rangle)\} + 1.$$
Normalizing winning strategies

Lemma
Let $\mathcal{T}$ be a finite transitive transition system and $\varphi \in \Sigma_2^{\mu}$. Suppose there is a winning strategy $\sigma$ for Player 0 in the parity game $\mathcal{E}(\varphi, (\mathcal{T}, s_0))$. If $y \in \text{bound}(\varphi)$ is a $\mu$-variable, then for every position $\langle y, s \rangle \in V|_\sigma$, we have that $\text{scc}(\langle y, s \rangle) = \emptyset$. 
Normalizing winning strategies

Given a winning strategy $\sigma$ for Player 0 on $E(\varphi, (T, s_0))$, the normalized (winning) strategy $\sigma^N$ for Player 0 is given by adapting $\sigma$ such that for all vertexes of the form $\langle \Diamond \beta, s' \rangle$ Player 0 moves to a vertex $\langle \beta, s'' \rangle$ whose measure is the minimal measure of all positions of the type $\langle \beta, \bar{s} \rangle$ reachable (in one step) from $\langle \Diamond \beta, s' \rangle$ in $E|_{\sigma}(\varphi, (T, s_0))$. 
Normalizing winning strategies

Theorem
Given a finite transitive transition system $T$, a formula $\varphi \in \Sigma_2^{\mu}$ such that all $\nu$-variables are weakly existential and a normalized winning strategy, $\sigma^N$, of Player 0 in $E(\varphi, (T, s))$. If in a play $\pi$ consistent with $\sigma^N$ there is a regeneration of a $\nu$-variable $x$ then

- either there is no more regeneration of a $\mu$-variable after the first regeneration of $x$
- or, if there is such a regeneration of a $\mu$-variable, then after this position there is no more regeneration of $x$. 
Another collapse

The collapse of the $\mu$-calculus over transitive models is then given by “coding” the previous Theorem via a translation

$$\rho \circ \tau : \Sigma_2^\mu \rightarrow \Delta_2^\mu$$

which preserves logical equivalence over transitive models.
Another collapse

\[ R : \mathcal{L}_\mu \rightarrow \Delta_2^{\mu} \] is defined as

- \[ R(p) = p \text{ and } R(\neg p) = \neg p \]
- \[ R(\bot) = \bot \text{ and } R(\top) = \top \]
- \[ R(\alpha \circ \beta) = R(\alpha) \circ R(\beta), \text{ where } \circ \in \{\land, \lor\} \]
- \[ R(\Box \beta) = \Box R(\beta), \text{ where } \Box \in \{\Box, \Diamond\} \]
- \[ R(\mu x. \varphi) = \text{nf}(\rho(\tau(\text{nf}(\mu x. (R(\varphi)))))) \]
- \[ R(\nu x. \varphi) = \neg (R(\mu x. \neg \varphi[x/\neg x])) \]
Another collapse

Theorem (AF (08a))

For all $\varphi \in \mathcal{L}_\mu$ and all transitive transition systems $T$ we have that

$$\|\varphi\|_T = \|R(\varphi)\|_T.$$
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

A final picture

The modal $\mu$-calculus over restricted classes of models: a final picture
### A final picture

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>strict</td>
<td>[AF(08)]</td>
<td>?</td>
<td>[AF(08)]: $\Delta^\mu_2$</td>
<td>[AF(08)]: $L_M$</td>
<td>[Vis(05)], [vBe(06)], [AF(08b)]: $L_M$</td>
</tr>
<tr>
<td>collapse</td>
<td>?</td>
<td>?</td>
<td>[AF(08)]: $L_M$</td>
<td>[AF(08)]: $L_M$</td>
<td>[AF(08b)]: $L_M$</td>
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The symmetric case

Conjecture

The modal $\mu$-calculus hierarchy is strict over symmetric models.
The symmetric case

Conjecture
The modal $\mu$-calculus hierarchy is strict over symmetric models.

Theorem (F(08))

The fixpoint alternation hierarchy of the modal $\mu$-calculus with backward modalities is strict over symmetric models.
Thank you!