Set Theoretic Geology

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Logic Colloquium 2008, 8 July 2008

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Definition

A transitive model *M* is a ground if it is a model of ZFC and there is a partial order $\mathbb{P} \in M$ and an *M*-generic filter $G \subseteq \mathbb{P}$ such that V = M[G].

Grounds

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Theorem (Laver)

If M is a ground, then M is a definable inner model.

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Grounds

Definition

A transitive model *M* is a ground if it is a model of ZFC and there is a partial order $\mathbb{P} \in M$ and an *M*-generic filter $G \subseteq \mathbb{P}$ such that V = M[G].

Theorem (Laver)

If M is a ground, then M is a definable inner model.

More precisely:

Theorem (Hamkins)

There is a formula $\varphi(x, y)$ such that whenever M is a ground of V, and M[G] = V, where $G \subseteq \mathbb{P} \in M$ is \mathbb{P} -generic, then, letting $\theta = \overline{\mathbb{P}}^+$,

$$\boldsymbol{M} = \{\boldsymbol{x} \mid \varphi(\boldsymbol{x}, \mathcal{P}(\theta) \cap \boldsymbol{M})\}.$$

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Applications

Lemma (Reitz)

The Ground Axiom, expressing that the universe has no non-trivial ground, is first order expressible.

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Applications

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Lemma (Reitz)

The Ground Axiom, expressing that the universe has no non-trivial ground, is first order expressible.

I have also made use of extensions of the uniform definability of grounds, in the context of maximality principles.

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Applications

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I have also made use of extensions of the uniform definability of grounds, in the context of maximality principles.

Some more applications are coming up, after motivating them.

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• Turn around the common direction of movement from grounds to forcing extensions.

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- Turn around the common direction of movement from grounds to forcing extensions.
- Strip away "random" information that was added by forcing.

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- Turn around the common direction of movement from grounds to forcing extensions.
- Strip away "random" information that was added by forcing.
- Find "canonical" models invariant for the forcing multiverse.

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- Turn around the common direction of movement from grounds to forcing extensions.
- Strip away "random" information that was added by forcing.
- Find "canonical" models invariant for the forcing multiverse.
- This is a new view of things, and there are many fundamental open questions!

The Mantle

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Definition

The Mantle \mathbb{M} is the intersection of all grounds.

G. Fuchs et al. (Münster/New York)

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The Mantle

Definition

The Mantle \mathbb{M} is the intersection of all grounds.

This mere definition is already an application of the uniform definability of grounds: The Mantle is a first order definable transitive class.

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A Question

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A Question

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Question

Is \mathbb{M} a model of ZF?

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A Question

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Question

Is \mathbb{M} a model of ZF? Of ZFC?

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Directedness

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Definition

The grounds are directed if whenever *M* and *N* are grounds, there is a ground *C* with $C \subseteq M \cap N$.

Directedness

Definition

The grounds are directed if whenever *M* and *N* are grounds, there is a ground *C* with $C \subseteq M \cap N$.

The grounds are set-directed if whenever $(W_x | x \in a)$ is a sequence of grounds, indexed by members of a set *a*, then there is a ground *C* with

$$C\subseteq \bigcap_{x\in a}W_x.$$

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Directedness

Definition

The grounds are directed if whenever *M* and *N* are grounds, there is a ground *C* with $C \subseteq M \cap N$.

The grounds are set-directed if whenever $(W_x | x \in a)$ is a sequence of grounds, indexed by members of a set *a*, then there is a ground *C* with

$$C\subseteq \bigcap_{x\in a}W_x$$

The grounds are locally set-directed if for every such sequence and every set *A*, there is a ground *C* such that

$$A\cap C\subseteq A\cap \bigcap_{x\in a}W_x.$$

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A Criterion

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Lemma

If the grounds are locally set-directed, then the Mantle is a model of ZFC.

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A Criterion

Lemma

If the grounds are locally set-directed, then the Mantle is a model of ZFC.

Question

Are the grounds directed? Set-directed? Locally set-directed?

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A Criterion

Lemma

If the grounds are locally set-directed, then the Mantle is a model of ZFC.

Question

Are the grounds directed? Set-directed? Locally set-directed?

Some partial answers will come later...

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Definition

The generic Mantle, gM, is the intersection of all grounds of all forcing extensions.

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Definition

The generic Mantle, gM, is the intersection of all grounds of all forcing extensions.

Note: It follows that

$$\mathsf{g}\mathbb{M} = \bigcap_{\alpha < \infty} \mathbb{M}^{\mathsf{V}^{\mathsf{Col}(\omega, \alpha)}}$$

G. Fuchs et al. (Münster/New York)

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The generic Mantle, gM, is the intersection of all grounds of all forcing extensions.

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So the more formal definition would be:

$$g\mathbb{M} = \{ \boldsymbol{x} \mid \forall \alpha \quad \operatorname{Col}(\omega, \alpha) \Vdash \check{\boldsymbol{x}} \in \mathbb{M} \}.$$

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We came up with this model because...

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Theorem

$$g\mathbb{M}\models ZF$$

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Theorem

$g\mathbb{M}\models \mathsf{ZF}$

The point is that $g\mathbb{M}$ is the same in every ground of every forcing extension of V.

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Theorem

$g\mathbb{M}\models \mathsf{ZF}$

The point is that $g\mathbb{M}$ is the same in every ground of every forcing extension of V.

In fact,

Theorem

The generic mantle is constant across the forcing multiverse.

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The generic HOD, gHOD, is the intersection of all HODs of all generic extensions.

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I introduced this model in connection with research on closed maximality principles. The point is:

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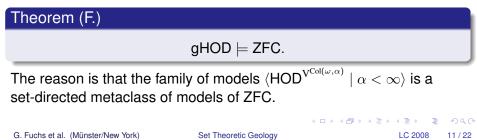
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Even more Models

Definition

The generic HOD, gHOD, is the intersection of all HODs of all generic extensions.

It follows that

$$\mathsf{gHOD} = igcap_{lpha < \infty} \mathsf{HOD}^{\mathrm{V^{Col}(\omega, lpha)}}$$

I introduced this model in connection with research on closed maximality principles. The point is:

Theorem (F.)

$$\mathsf{gHOD} \models \mathsf{ZFC}.$$

The reason is that the family of models $\langle HOD^{V^{Col}(\omega,\alpha)} | \alpha < \infty \rangle$ is a set-directed metaclass of models of ZFC. Again, gHOD is constant across the forcing multiverse.

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A Question

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Question

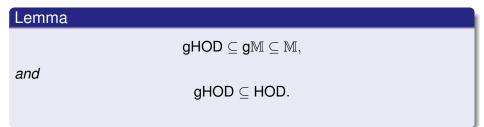
What is the relationship between $\mathbb{M},$ g \mathbb{M} and gHOD?

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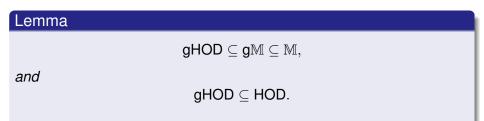
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Some answers



It is trivial that $gHOD \subseteq HOD$ and that $g\mathbb{M} \subseteq \mathbb{M}$, so the only inclusion of substance is that $gHOD \subseteq g\mathbb{M}$.

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Let V' = W[g] = V[h], for g generic over W and h generic over V.
 In other words, W is a generic ground.

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- Let V' = W[g] = V[h], for g generic over W and h generic over V.
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- Let θ be larger than the size of the forcings leading from W to V' and from V to V'.

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 In other words, W is a generic ground.
- Let θ be larger than the size of the forcings leading from W to V' and from V to V'.
- Let *I* be Col(ω, θ)-generic over V'. Then there is g'
 Col(ω, θ)-generic over W and h' Col(ω, θ)-generic over V, such that

$$W[g'] = W[g][l] = V'[l] = V[h][l] = V[h'].$$

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$$W[g'] = W[g][l] = V'[l] = V[h][l] = V[h'].$$

• By the homogeneity of the collapse, it follows that

$$\mathsf{HOD}^{V'[I]} = \mathsf{HOD}^{W[g']} \subseteq W.$$

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- Let V' = W[g] = V[h], for g generic over W and h generic over V.
 In other words, W is a generic ground.
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- Let *I* be Col(ω, θ)-generic over V'. Then there is g'
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$$W[g'] = W[g][l] = V'[l] = V[h][l] = V[h'].$$

By the homogeneity of the collapse, it follows that

$$\mathsf{HOD}^{\mathsf{V}'[I]} = \mathsf{HOD}^{W[g']} \subseteq W.$$

• So
$$gHOD = \bigcap_{\alpha < \infty} HOD^{V^{Col(\omega, \alpha)}} \subseteq g\mathbb{M}$$
, as claimed.

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When V is constructible from a set

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When V is constructible from a set

Theorem

If the universe is constructible from a set, then the grounds are set-directed, so the Mantle is a model of ZFC.

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When V is constructible from a set

Theorem

If the universe is constructible from a set, then the grounds are set-directed, so the Mantle is a model of ZFC.

Theorem

If the universe is constructible from a set, then

 $gHOD = g\mathbb{M} = \mathbb{M}.$

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As a reminder: In general,

$gHOD\subseteq g\mathbb{M}\subseteq \mathbb{M}, and \ gHOD\subseteq HOD.$

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As a reminder: In general,

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Theorem

Fix a model V of ZFC.

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As a reminder: In general,

$gHOD \subseteq g\mathbb{M} \subseteq \mathbb{M}$, and $gHOD \subseteq HOD$.

Theorem

Fix a model V of ZFC. Then V has proper class forcing extensions N_0 , N_1 , N_2 , N_3 such that:

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$$V = gHOD^{N_0} = gM^{N_0} = M^{N_0} = HOD^{N_0} \subsetneqq N_0$$

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.

•
$$V = gHOD^{N_2} = HOD^{N_2} \subseteq \mathbb{M}^{N_2} = N_2.$$

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Fix a model V of ZFC. Then V has proper class forcing extensions N_0 , N_1 , N_2 , N_3 such that:

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• $V = gHOD^{N_1} = gM^{N_1} = M^{N_1} \subsetneqq HOD^{N_1} = N_1.$

•
$$V = gHOD^{N_2} = HOD^{N_2} \subsetneq M^{N_2} = N_2.$$

• $V \subsetneq gHOD^{N_3} = g\mathbb{M}^{N_3} = \mathbb{M}^{N_3} = HOD^{N_3} = N_3.$

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$$V = gHOD^{N_0} = g\mathbb{M}^{N_0} = \mathbb{M}^{N_0} = HOD^{N_0} \subsetneqq N_0$$

• Let $\delta_{\alpha} = \beth_{\omega(\alpha+1)}^+$.

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$$V = gHOD^{N_0} = g\mathbb{M}^{N_0} = \mathbb{M}^{N_0} = HOD^{N_0} \subsetneqq N_0$$

- Let $\delta_{\alpha} = \beth_{\omega(\alpha+1)}^+$.
- Force with P = Π_{α<∞} Q_α, (with set support) where Q_α is the lottery sum of:

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$$\mathrm{V} = \mathsf{gHOD}^{N_0} = \mathsf{gM}^{N_0} = \mathbb{M}^{N_0} = \mathsf{HOD}^{N_0} \subsetneqq N_0$$

- Let $\delta_{\alpha} = \beth_{\omega(\alpha+1)}^+$.
- Force with $\mathbb{P} = \prod_{\alpha < \infty} \mathbb{Q}_{\alpha}$, (with set support) where \mathbb{Q}_{α} is the lottery sum of:
 - a forcing that forces GCH at δ_{α} , namely Add(δ_{α}^+ , 1), or

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 - a forcing that destroys the GCH at δ_{α} , namely $\operatorname{Add}(\delta_{\alpha}, (2^{<\delta_{\alpha}})^{++})$.

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 - a forcing that forces GCH at δ_{α} , namely Add $(\delta_{\alpha}^+, 1)$, or
 - a forcing that destroys the GCH at δ_{α} , namely $Add(\delta_{\alpha}, (2^{<\delta_{\alpha}})^{++})$.
- Let *G* be \mathbb{P} -generic, and let $N_0 = V[G]$.

- Let $\delta_{\alpha} = \beth_{\omega(\alpha+1)}^+$.
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 - a forcing that destroys the GCH at δ_{α} , namely $\operatorname{Add}(\delta_{\alpha}, (2^{<\delta_{\alpha}})^{++})$.
- Let *G* be \mathbb{P} -generic, and let $N_0 = V[G]$.
- A density argument shows that in the forcing extension, every set of V is coded into the GCH-pattern, and hence is in HOD^{V[G]}.

- Let $\delta_{\alpha} = \beth_{\omega(\alpha+1)}^+$.
- Force with $\mathbb{P} = \prod_{\alpha < \infty} \mathbb{Q}_{\alpha}$, (with set support) where \mathbb{Q}_{α} is the lottery sum of:
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 - a forcing that destroys the GCH at δ_{α} , namely $\operatorname{Add}(\delta_{\alpha}, (2^{<\delta_{\alpha}})^{++})$.
- Let *G* be \mathbb{P} -generic, and let $N_0 = V[G]$.
- A density argument shows that in the forcing extension, every set of V is coded into the GCH-pattern, and hence is in HOD^{V[G]}.
- In fact, the same argument shows that P forces that every set of V is coded in the GCH-pattern unboundedly often, so that V ⊆ gHOD^{V[G]}.

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.

- Force with $\mathbb{P} = \prod_{\alpha < \infty} \mathbb{Q}_{\alpha}$, (with set support) where \mathbb{Q}_{α} is the lottery sum of:
 - a forcing that forces GCH at δ_{α} , namely Add(δ_{α}^{+} , 1), or
 - a forcing that destroys the GCH at δ_{α} , namely $Add(\delta_{\alpha}, (2^{<\delta_{\alpha}})^{++})$.
- Let *G* be \mathbb{P} -generic, and let $N_0 = V[G]$.
- A density argument shows that in the forcing extension, every set of V is coded into the GCH-pattern, and hence is in HOD^{V[G]}.
- In fact, the same argument shows that P forces that every set of V is coded in the GCH-pattern unboundedly often, so that V ⊆ gHOD^{V[G]}.
- So far we know that $V \subseteq gHOD^{V[G]} \subseteq g\mathbb{M}^{V[G]} \subseteq \mathbb{M}^{V[G]}$, and that $V \subseteq HOD^{V[G]}$.

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• Let $x \in \mathbb{M}^{\mathbb{V}[G]}$. $x = \tau^{G}$ for a name τ which is a set, so $x \in \mathbb{V}[G_{\alpha}]$, for some α .

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- So $\mathbb{M}^{\mathbb{V}[G]} \subseteq \mathbb{V}[G^{\alpha}].$
- So $x \in V[G^{\alpha}] \cap V[G_{\alpha}]$.
- But these are mutually generic filters, so $x \in V$.

$\mathsf{HOD}^{V[G]} \subseteq V$

• Fix $\alpha < \infty$.

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- So $HOD^{V[G]} = HOD^{V[G^{\alpha}][G_{\alpha}]} \subseteq V[G^{\alpha}].$
- This is true for every α .
- But we have already seen in the previous argument that $\bigcap_{\alpha < \infty} V[G^{\alpha}] = V.$

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- Also, every member of V is coded in the continuum function unboundedly often, so that V ⊆ gHOD^{V[G]}.
- So far, we have:

$$V \subseteq \mathsf{gHOD}^{V[G]} \subseteq \mathsf{gM}^{V[G]} \subseteq \mathbb{M}^{V[G]}, \mathsf{and} \ V[G] = \mathsf{HOD}^{V[G]}$$

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$\mathbb{M}^{N_1} \subseteq V$

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• By the product analysis: Every $V[G^{\alpha}]$ is a ground of N_1 .

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$\mathbb{M}^{\textit{N}_1} \subseteq V$

By the product analysis: Every V[G^α] is a ground of N₁.
So M^{N₁} ⊆ ∩_{α≤∞} V[G^α] = V.

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Question

Is gM = M?

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Question

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