Set Theoretic Geology

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Grounds

Definition

A transitive model M is a ground if it is a model of ZFC and there is a partial order $\mathbb{P} \in M$ and an M-generic filter $G \subseteq \mathbb{P}$ such that $\mathrm{V} = M[G]$.

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Theorem (Laver)

If M is a ground, then M is a definable inner model.

More precisely:

Theorem (Hamkins)

There is a formula $\varphi(x, y)$ such that whenever M is a ground of V, and $M[G] = \mathrm{V}$, where $G \subseteq \mathbb{P} \in M$ is $\mathbb{P}$-generic, then, letting $\theta = \overline{\overline{\mathbb{P}}}^{+}$,

$$M = \{x \mid \varphi(x, \mathcal{P}(\theta) \cap M)\}.$$

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I have also made use of extensions of the uniform definability of grounds, in the context of maximality principles.
Some more applications are coming up, after motivating them.

Motivation

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- Strip away “random” information that was added by forcing.
- Find “canonical” models invariant for the forcing multiverse.
- This is a new view of things, and there are many fundamental open questions!

The Mantle

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This mere definition is already an application of the uniform definability of grounds: The Mantle is a first order definable transitive class.

A Question

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The grounds are locally set-directed if for every such sequence and every set A, there is a ground C such that

$$A \cap C \subseteq A \cap \bigcap_{x \in a} W_x.$$

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Some partial answers will come later...

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We came up with this model because...

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In fact,

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The generic mantle is constant across the forcing multiverse.

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The reason is that the family of models $\langle \mathrm{HOD}^{V^{\mathrm{Col}(\omega,\alpha)}} \mid \alpha < \infty\rangle$ is a set-directed metaclass of models of ZFC. Again, gHOD is constant across the forcing multiverse.

A Question

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What is the relationship between $\mathbb{M}$, g$\mathbb{M}$ and gHOD?

Some answers

Lemma

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It is trivial that $\mathrm{gHOD} \subseteq \mathrm{HOD}$ and that $\mathrm{g}\mathbb{M} \subseteq \mathbb{M}$, so the only inclusion of substance is that $\mathrm{gHOD} \subseteq \mathrm{g}\mathbb{M}$.

gHOD $\subseteq$ g$\mathbb{M}$

- Let $\mathrm{V}' = W[g] = \mathrm{V}[h]$, for g generic over W and h generic over V. In other words, W is a generic ground.

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- Let θ be larger than the size of the forcings leading from W to V' and from V to V'.
- Let I be $\mathrm{Col}(\omega, \theta)$-generic over V'. Then there is g' $\mathrm{Col}(\omega, \theta)$-generic over W and h' $\mathrm{Col}(\omega, \theta)$-generic over V, such that
$$W[g'] = W[g][I] = \mathrm{V}'[I] = \mathrm{V}[h][I] = \mathrm{V}[h'].$$

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$$W[g'] = W[g][I] = \mathrm{V}'[I] = \mathrm{V}[h][I] = \mathrm{V}[h'].$$
- By the homogeneity of the collapse, it follows that
$$\mathrm{HOD}^{\mathrm{V}'[I]} = \mathrm{HOD}^{W[g']} \subseteq W.$$

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- By the homogeneity of the collapse, it follows that
$$\mathrm{HOD}^{\mathrm{V}'[I]} = \mathrm{HOD}^{W[g']} \subseteq W.$$
- So gHOD $= \bigcap_{\alpha<\infty} \mathrm{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}} \subseteq$ g$\mathbb{M}$, as claimed.

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More answers

As a reminder: In general,

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- So far we know that $\mathrm{V} \subseteq \mathrm{gHOD}^{\mathrm{V}[G]} \subseteq \mathrm{g}\mathbb{M}^{\mathrm{V}[G]} \subseteq \mathbb{M}^{\mathrm{V}[G]}$, and that $\mathrm{V} \subseteq \mathrm{HOD}^{\mathrm{V}[G]}$.

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- Let $x \in \mathbb{M}^{\mathrm{V}[G]}$. $x = \tau^G$ for a name τ which is a set, so $x \in \mathrm{V}[G_\alpha]$, for some α.

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- $\mathrm{V}[G] = \mathrm{V}[G_\alpha][G^\alpha]$, where G_α is $\mathbb{P}_\alpha$-generic, which is a set sized forcing. So $\mathrm{V}[G^\alpha]$ is a ground of $\mathrm{V}[G]$.

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- So $\mathbb{M}^{\mathrm{V}[G]} \subseteq \mathrm{V}[G^\alpha]$.
- So $x \in \mathrm{V}[G^\alpha] \cap \mathrm{V}[G_\alpha]$.
- But these are mutually generic filters, so $x \in \mathrm{V}$.

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- Fix $\alpha < \infty$.

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- Fix $\alpha < \infty$.
- $\mathbb{P}_\alpha$ is not weakly homogeneous.
- But it is "densely weakly homogeneous", in the sense that the set of $p \in \mathbb{P}_\alpha$ such that $\mathbb{P}_\alpha(\leq p)$ is weakly homogeneous is dense in $\mathbb{P}_\alpha$.

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- So $\mathrm{HOD}^{\mathrm{V}[G]} = \mathrm{HOD}^{\mathrm{V}[G^\alpha][G_\alpha]} \subseteq \mathrm{V}[G^\alpha]$.
- This is true for every α.
- But we have already seen in the previous argument that $\bigcap_{\alpha<\infty} \mathrm{V}[G^\alpha] = \mathrm{V}$.

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- So far, we have:

$$\mathrm{V} \subseteq \mathrm{gHOD}^{\mathrm{V}[G]} \subseteq \mathrm{g}\mathbb{M}^{\mathrm{V}[G]} \subseteq \mathbb{M}^{\mathrm{V}[G]}, \text{ and } \mathrm{V}[G] = \mathrm{HOD}^{\mathrm{V}[G]}.$$

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- So $\mathbb{M}^{N_1} \subseteq \bigcap_{\alpha<\infty} \mathrm{V}[G^\alpha] = \mathrm{V}$.

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What is $\mathrm{g}\mathbb{M}^{N_2}$?
Is it geology or archeology?