On Standard SBL-Algebras with Added Involutive Negations

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Hájek's BL

Petr Hájek: *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, 1998.

Introduces the Basic Fuzzy Logic BL

Intended semantics: algebras given by continuous t-norms on $[0, 1]$

BL — Language, Syntax

Basic connectives: $\&$, $\to$, 0

Definable connectives:

$$\begin{aligned}
\neg\varphi &\text{ is } \varphi \to 0 \\
\varphi \wedge \psi &\text{ is } \varphi \& (\varphi \to \psi) \\
\varphi \vee \psi &\text{ is } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\varphi \equiv \psi &\text{ is } (\varphi \to \psi) \& (\psi \to \varphi) \\
1 &\text{ is } 0 \to 0
\end{aligned}$$

Syntax: classical

BL — Standard Semantics

A t-norm $*$ is a binary operation on $[0,1]$ such that:

- $*$ is commutative and associative
- $*$ is non-decreasing in both arguments
- $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

The residuum $\Rightarrow$ of a *continuous* t-norm $*$ is
$x \Rightarrow y = \max\{z \mid x * z \leq y\}$.

The standard algebra determined by $*$ on $[0,1]$ is $\langle [0,1], *, \Rightarrow, 0 \rangle$.

Evaluation of BL-formulas: $*$ interprets & and $\Rightarrow$ interprets $\rightarrow$.

Examples and characterization

Important continuous t-norms:

- Łukasiewicz t-norm: $x * y$ is $\max(x + y - 1, 0)$
- Gödel t-norm: $x * y$ is $\min(x, y)$
- product t-norm: $x * y$ is $x.y$

Mostert-Shields theorem: Each continuous t-norm is an "ordinal sum" of isomorphic copies of Łukasiewicz, Gödel, and product t-norms.

SBL with Involutive Negation

SBL – logic of continuous t-norms with strict definable negation

New connective: involutive negation $\sim$

Semantics: decreasing involution on $[0, 1]$, i.e.,

- $x < y$ implies $\sim y <\sim x$ for all $x, y \in [0, 1]$
- $\sim\sim x = x$ for all $x \in [0, 1]$.

Example: $1 - x$

Algebras: $\langle [0, 1], *, \Rightarrow, 0, \sim \rangle$, shortly $\langle *, \sim \rangle$

Types of algebras considered

Algebras $\langle *, \sim \rangle$.

Continuous t-norm $*$ which

- has the strict negation
- is finite ordinal sum of Ł- and Π-components.

A finite ordinal sum of L- and Π-components is an algebra $C_1 \oplus \ldots \oplus C_n$, $n \in N$, each $C_i = L$ or $C_i = \Pi$.

Arbitrary involutive negation.

When are two involutive negations isomorphic?

Definition

Let $*$ be a continuous t-norm and $\sim_1$, $\sim_2$ two involutive negations. Then $\sim_1$ and $\sim_2$ are isomorphic w. r. t. $*$ iff $\langle *, \sim_1 \rangle$ is isomorphic to $\langle *, \sim_2 \rangle$.

Any such isomorphism is an automorphism of $*$ on $[0, 1]$.

Automorphisms of continuous t-norms

Lemma

$f : [0,1] \longrightarrow [0,1]$ *is an automorphism of* $*$ *iff*

- $*$ *is* Π*:* $f(x) = x^r$ *for some real* $r > 0$ *(Hion's Lemma)*
- $*$ *is Ł:* f *is an identity on* $[0,1]$ *(C., d'O., M.)*
- $*$ *is a finite sum of Ł's and* Π*'s:*
  f *is identity on Ł-components;*
  f *is an r-power w. r. t.* $*$ *on* Π*-components*

Problem

Characterize all continuous t-norms $*$ for which the equivalence

$$\mathrm{TAUT}(\langle *, \sim_1\rangle) = \mathrm{TAUT}(\langle *, \sim_2\rangle)$$

iff

$$\langle *, \sim_1\rangle \text{ is isomorphic to } \langle *, \sim_2\rangle$$

holds for arbitrary involutive negations $\sim_1$ and $\sim_2$.

Additionally, if for $*$, $\sim_1$, $\sim_2$

$$\mathrm{TAUT}(\langle *, \sim_1\rangle) \neq \mathrm{TAUT}(\langle *, \sim_2\rangle),$$

are these sets comparable by inclusion?

Characterizing theorem

Theorem

Let $*$ be a finite ordinal sum of Ł- and Π-components, where the first component is Π. Then

(i) If $*$ is Π, $\Pi \oplus j.\text{Ł}$, or $\Pi \oplus i.\text{Ł} \oplus \Pi \oplus j.\text{Ł}$, for $i \geq 0$, $j > 0$, then non-isomorphic negations yield distinct and incomparable sets of tautologies.

(ii) Otherwise (if $*$ is of type $\Pi \oplus i.\text{Ł} \oplus \Pi$ or it contains at least three product components), there are two non-isomorphic negations yielding the same set of tautologies.

If the sets of tautologies are distinct, they are also incomparable by inclusion

Tractability of Involutive Negations

Task: describe the graph of $\sim$ by a family of propositional formulas

Method:

- find a dense definable set S in $[0, 1]$
- compare the values of $\sim$ on S against the values in S

Note: formulas defining S may contain $\sim$

Product t-norm

Assume $\sim_1$ and $\sim_2$ non-isomorphic.

Taking possibly isomorphic copies, make $0 < a < 1$ the fixed point of $\sim_1$ and $\sim_2$.

For i, j, r, s positive integers, compare $\sim a^{i/j}$ against $a^{r/s}$.
To do so, consider the family of formulas

$$\Phi(i/j, r/s) \text{ is } \Delta(q \equiv \sim q) \& \Delta(z^j \equiv q^i) \to \Delta(q^r \to (\sim z)^s)$$
$$\Phi'(i/j, r/s) \text{ is } \Delta(q \equiv \sim q) \& \Delta(z^j \equiv q^i) \to \Delta((\sim z)^s \to q^r)$$

Product t-norm (cont.)

Theorem

If $\langle *, \sim_1 \rangle$ is not isomorphic to $\langle *, \sim_2 \rangle$, then TAUT$\langle *, \sim_1 \rangle$ and TAUT$\langle *, \sim_2 \rangle$ are distinct and incomparable.

Other t-norms with distinguishable negations

Types of ordinal sum ($i \geq 0$, $j > 0$): $\Pi \oplus j.\text{Ł}$ or $\Pi \oplus i.\text{Ł} \oplus \Pi \oplus j.\text{Ł}$.

Assume $\sim_1$ and $\sim_2$ non-isomorphic.

Idempotent elements of $*$ are definable by formulas without $*$.

In Ł-components, dense sets of values are definable by formulas without $\sim$.

To define dense sets of values in Π-components,

- use the fixed point, or
- map values in Ł-components into Π-components using $\sim$ (taking possibly isomorphic copies of $\sim_1$ or $\sim_2$).

Other t-norms with distinguishable negations (cont.)

Lemma

Let $x, y \in [0,1]$ be definable for $\sim_1$, $\sim_2$. Assume $\sim_1 x < \sim_2 x$ and $\sim_1 x \leq y \leq \sim_2 x$. Then, $TAUT(\langle *, \sim_1\rangle)$ and $TAUT(\langle *, \sim_2\rangle)$ are incomparable.

Distinguishing formulas (example):

$$\Delta[\phi(\overline{x})\,\&\,\psi(\overline{y})] \longrightarrow \Delta(\sim \overline{x} \to \overline{y})$$

$$\Delta[\phi(\overline{x})\,\&\,\psi(\overline{y})] \longrightarrow \Delta(\overline{y} \to \sim \overline{x})\,\&\,\neg\Delta(\sim \overline{x} \to \overline{y})$$

T-norms with indistinguishable negations

Assume $*$ is s. t. there is an involutive negation which maps two product components of $*$ onto each other.

This is iff $*$ is of type $\Pi \oplus i.\text{Ł} \oplus \Pi$ or it contains at least three product components.

Theorem

If $*$ is as above, there are two non-isomorphic involutive negations $\sim_1$ and $\sim_2$ yielding the same sets of tautologies.

Open problems

- Find axiomatizations for some of the sets of tautologies.
- Determine the complexity of some of the sets of tautologies.
- Consider (some types of) infinite sums.

Related results

Cintula P., Klement E. P., Mesiar R., Navara M.: Residuated logics based on strict triangular norms with an involutive negation, **MLQ** 52, 2006.
Investigates lattice of subvarieties of SBL$\sim$; for $\Pi\sim$, the lattice is of infinite height and width.

Gehrke M., Walker C., Walker E: Fuzzy logics arising from strict de Morgan systems, **Topological and Algebraic Structures in Fuzzy Sets**, Kluwer 2003.
In a language without $\Rightarrow$, each two non-isomorphic algebras given by product t-norm $*$ and by $\sim$ differ in the sets of valid identities.