

Generalised Hrushovski constructions

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Outline of the talk

1. Introduction
2. Generalised free fusion
3. A variant: bicoloured fields and bad fields
4. Generic automorphisms of Hrushovski constructions

The geometry of strongly minimal sets

- ▶ A **pregeometry** on a set X is given by a finitary closure operator $\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfying the exchange lemma.
- ▶ Get notions of **dimension**, **independence**, **basis** etc.
- ▶ Ex: linear independence in a v.s.; alg. independence in a field.

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Trichotomy Conjecture (B. Zilber 1980)

Let T be a strongly minimal theory. Then, there are three cases:

- ▶ The geometry of T is **trivial**.
- ▶ T has a **locally modular** non-trivial geometry (projective or affine geometry over some skew-field).
- ▶ If T is not locally modular, T **interprets an algebraically closed field**.

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- ▶ most noteworthy: the **fusion construction** by Hrushovski, showing e.g. that there is a strongly minimal structure $(M, +_1, \cdot_1, +_2, \cdot_2)$ such that, for $i = 1, 2$, $(M, +_i, \cdot_i) \models ACF_{p_i}$.

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Theorem (E.Hrushovski 1992)

Let T_1 et T_2 be strongly minimal theories in (countable) disjoint languages, with definable multiplicities (DMP). Then, there is a strongly minimal theory $T \supseteq T_1 \cup T_2$.

Strongly minimal fusion: two steps

free fusion (we are mainly interested in this part in our talk):

- ▶ *Predimension function* (on finite subsets of $\mathcal{L}_1 \cup \mathcal{L}_2$ -structures)

$$\delta(A) := \dim_1(A) + \dim_2(A) - |A|;$$

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- ▶ put $\mathcal{C} := \{A \text{ finite} \mid \emptyset \leq A\}$;
- ▶ (\mathcal{C}, \leq) is countable, has (AP), (JEP) and (HP);
- ▶ the Fraïssé limit of (\mathcal{C}, \leq) is ω -stable of rank ω .

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collapse:

Amalgamate in a restricted class, uniformly bounding the number of solutions of sets of dim. 0 \Rightarrow get a strongly minimal theory.

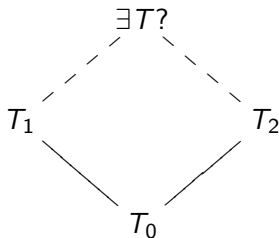
Question (Hrushovski 1992)

Let T_1 and T_2 be s.m. (in countable languages with DMP) which intersect in an infinite vector space over a finite field. Is it possible to find a s.m. completion of $T_1 \cup T_2$?

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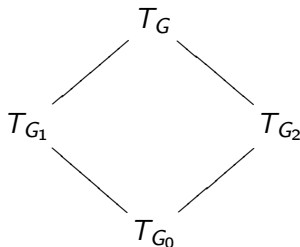
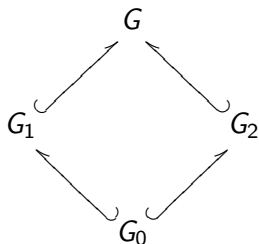
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More generally: Is it possible to find a s.m. fusion T of two s.m. theories T_1, T_2 intersecting in some third theory T_0 ?



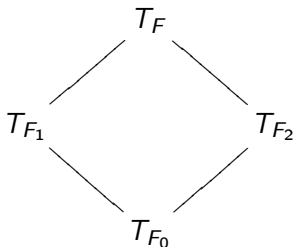
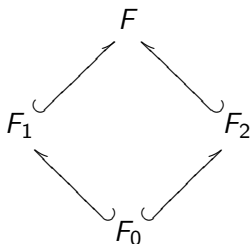
A trivial example

- ▶ For G a group let $T_G =$ **theory of an infinite free G -action**.
($\Rightarrow T_G$ trivial strongly minimal)
- ▶ $G_0 \leq G_1, G_2, G := G_1 *_{G_0} G_2$
 $\Rightarrow T_G$ strongly minimal fusion of T_{G_1} and T_{G_2} over T_{G_0} .



A modular non-trivial example

- ▶ Let T_F be the **theory of an infinite vector space over F** , where F is a skew-field. ($\Rightarrow T_F$ modular strongly minimal)
- ▶ For $F_0 \subseteq F_1, F_2$, the ring $F_1 *_{F_0} F_2$ allows a field of fractions $F \Rightarrow T_F$ strongly minimal fusion of T_{F_1} and T_{F_2} over T_{F_0} .



Two non-examples, two obstructions

1. Consider the following relative fusion context:

- ▶ $T_0 = \mathbb{Q}$ -vector spaces, with c, d (linearly independent) named,
- ▶ $T_1 = \mathbb{Q}[i]$ -vector spaces, with $i \cdot c = d$,
- ▶ $T_2 = \mathbb{Q}(X)$ -vector spaces, with $X \cdot c = d$.

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2. There is an example with the following properties:

- ▶ T_1, T_2 are modular s.m., T_0 trivial and ω -categorical;
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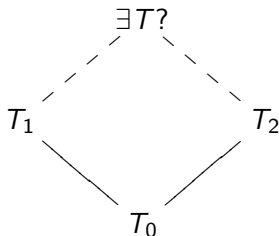
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\Rightarrow obstructions to a s.m. fusion, a **logical** and a **geometrical** one:

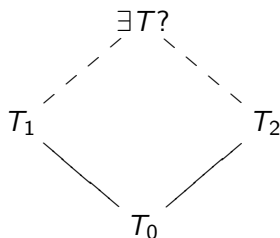
- ▶ Definability problems if T_0 is not ω -categorical.
- ▶ The geometrical interaction of the two structures can be *wild*.

Weaker requirements on the fusion theory



Weaken the requirements on T : it should support a fusion of the pregeometries, in a model-theoretically meaningful way, e.g.

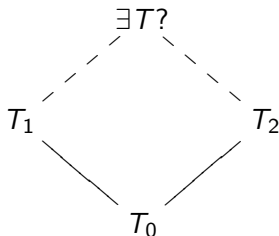
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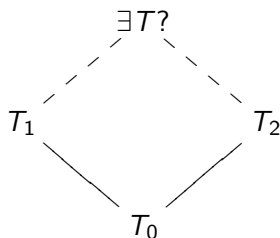
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- ▶ as the forking pregeometry attached to a regular type, where T is stable (ex.: free fusion in the original context);
- ▶ as before, with "simple" instead of "stable";
- ▶ as a pregeometry coming from some independence relation.

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- ▶ If in addition T eliminates \exists^∞ , it is called **geometric**.
- ▶ Examples of geometric theories:
 - ▶ Strongly minimal theories, more generally simple theories of SU-rank 1 (e.g. the random graph, pseudofinite fields).
 - ▶ *RCF*, more generally *o*-minimal theories.
 - ▶ $\text{Th}(\mathbb{Q}_p)$, as well as *ACVF*.
 - ▶ Reducts of (pre-)geometric theories are (pre-)geometric.

A context for the construction

- Assumptions:
- T_1, T_2 are pregeometric theories;
 - T_0 is strongly minimal and modular.
- ▶ Work with the predimension $\delta = \dim_1 + \dim_2 - \dim_0$.
- ▶ Obtain a *fusion class* (\mathcal{C}, \leq) . Structures in \mathcal{C} are finitely $\langle \cdot \rangle$ -generated ($\langle \cdot \rangle =$ transitive closure of acl_1 and acl_2).

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 - ▶ (\mathcal{C}, \leq) has (AP) \Rightarrow rich structures do exist.
Let T_ω be the theory of all rich structures.
 - ▶ In general, (\mathcal{C}, \leq) does not have (JEP) $\Rightarrow T_\omega$ incomplete.
 - ▶ saturated models of T_ω are not necessarily rich.

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Definability assumptions:

- T_1, T_2 are geometric (pregeometric with elimination of \exists^∞)
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Theorem

In this context, T_ω can be axiomatised. We obtain:

1. *Sufficiently saturated models of T_ω are rich.*
2. *In T_ω , every formula is equivalent to a boolean combinations of bounded existential formulas (assuming the T_i have QE in \mathcal{L}_i).*
3. *The completions of T_ω are determined by $\text{qftp}_{\mathcal{L}}(\langle\emptyset\rangle)$ (here, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$).*

- Context:
- T_1, T_2 simple SU-rank 1 (in particular geometric)
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 - The expansions $T_i \supseteq T_0$ satisfy condition **A** for $i = 1, 2$.

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- ▶ The proof uses the Theorem of Kim-Pillay. Difficult to establish: the *Independence Theorem*.
- ▶ Note the similarities with results of Chatzidakis-Pillay about stable theories with a generic automorphism (I will come to that later).

Fact

In the following cases, $T_1 \supseteq T_0$ satisfies condition **A**:

1. T_0 with trivial pregeometry, T_1 arbitrary
2. T_1 strongly minimal, T_0 arbitrary.
3. F a pseudofinite field, $T_1 = \text{Th}(F, +, \times)$, $T_0 = \text{Th}(F, +)$

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Corollary

1. Two arbitrary SU-rank 1 theories can be fused into a simple theory of SU-rank $\leq \omega$.
2. For ω -categorical T_0 , and s.m. expansions $T_1, T_2 \supseteq T_0$, there is a simple fusion of T_1 and T_2 over T_0 (of SU-rank $\leq \omega$).
3. There is a simple structure $(F, +, \times_1, \times_2)$ of SU-rank ω such that $(F, +, \times_1) \models \text{PSF}_p$ and $(F, +, \times_2) \models \text{ACF}_p$ (for $p > 0$).

- Setting:
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Theorem (Hasson, H. 2006)

In the above setting, T_ω is complete ω -stable with a unique generic type of rank ω .

A detailed description of the types to be collapsed can be given.

Strongly minimal fusion: collapse results

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- ▶ Baudisch, Martin Pizarro, Ziegler 2007: for $T_0 =$ theory of an infinite \mathbb{F}_q -vector space;
- ▶ Not hard to see: the arbitrary case reduces to one of the previous cases.

A variant: Poizat's bicoloured fields I

Poizat (1999,2001): Construction of various expansions of algebraically closed fields K (adding a new predicate).

Black fields:

- ▶ $N^K \subseteq K$ distinguished *subset*, $\text{char}(K)$ arbitrary (but fixed).
- ▶ Predimension $\delta((K, N^K)) = 2 \text{tr. deg}(K) - |N^K|$
- ▶ Free amalgamation \Rightarrow black field of Morley rank $\omega \cdot 2$.
- ▶ Collapse (Poizat, Baldwin-Holland) to a black field of MR 2.

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Red fields:

- ▶ $R^K \subseteq K$ distinguished *additive subgroup*, $\text{char}(K) = p > 0$.
- ▶ Predimension $\delta((K, R^K)) = 2 \text{tr. deg}(K) - \text{l. dim}_{\mathbb{F}_p}(R^K)$
- ▶ Free amalgamation \Rightarrow red field of Morley rank $\omega \cdot 2$.
- ▶ Collapse (Baudisch, Martin Pizarro, Ziegler) to MR 2.

Green fields:

- ▶ $\ddot{U}^K \subseteq K^*$ is a distinguished *multiplicative subgroup*, with \ddot{U} divisible and torsion free, $\text{char}(K) = 0$.
- ▶ Predimension $\delta((K, \ddot{U}^K)) = 2 \text{tr. deg}(K) - \text{l. dim}_{\mathbb{Q}}(\ddot{U}^K)$
- ▶ Free amalgamation \Rightarrow green field of Morley rank $\omega \cdot 2$, the subgroup \ddot{U} is of rank ω .

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Remark

Bicoloured structures have simple analogues, e.g. for $p > 0$, there is $F \models \text{Psf}_p$ with an additive subgroup $R^F \leq F$ such that (F, R) is supersimple of SU-rank $\omega \cdot 2$.

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Every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.

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- ▶ An obstacle in the initial proof strategy:

Bad Fields: structures (K, \ddot{U}) of finite Morley rank with $K \models ACF$ and \ddot{U} a proper infinite subgroup of (K^*, \cdot) .

- ▶ Longstanding open question of B. Zilber: Do bad fields exist?

Algebraicity Conjecture (Cherlin-Zilber)

Every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.

- ▶ An obstacle in the initial proof strategy:

Bad Fields: structures (K, \ddot{U}) of finite Morley rank with $K \models \text{ACF}$ and \ddot{U} a proper infinite subgroup of (K^*, \cdot) .

- ▶ Longstanding open question of B. Zilber: Do bad fields exist?

Theorem (Baudisch, Martin Pizarro, H., Wagner)

There is a bad field (K, \ddot{U}) in char. 0, obtained by collapsing Poizat's green field ("bad field of infinite rank").

Generic automorphisms of stable theories

- ▶ Let T be a stable, complete and model-complete \mathcal{L} -theory.
(in case T is not model-complete, we Morleyise first)
- ▶ $(M, \sigma) \models T_\sigma$ iff $M \models T$ and $\sigma \in \text{Aut}_{\mathcal{L}}(M)$ (in $\mathcal{L} \cup \{\sigma\}$)
- ▶ We say the *generic automorphism is axiomatisable in T*
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Fact (Chatzidakis-Pillay)

TA is a simple theory. Every formula is equivalent to a boolean combination of bounded existential formulas (assuming T has QE), and its completions are given by the action of σ on $\text{acl}(\emptyset)$

Theorem

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 - ▶ *Hrushovski's ab initio construction;*
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Fact (Hasson-Hrushovski)

For strongly minimal T , the generic automorphism is axiomatisable iff T has the DMP.

Idea of the proof.

We first establish a general criterion: The generic automorphism is axiomatisable if we have a *notion of genericity* s.t.

- ▶ there are "enough" formulas containing a single generic type;
- ▶ "containing a single generic type" is definable in families;
- ▶ "projecting the generic on the generic" is definable in families.

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Remark

In the corresponding collapsed versions, the axiomatisability of the generic automorphism follows from results of Chatzidakis-Pillay. It can also be shown using the above general criterion.

Compatibility with other generic constructions: lovely pairs

- ▶ Ben-Yaacov, Pillay and Vassiliev introduced **Lovely pairs** (of models) of a simple theory T , i.e. $\mathcal{L} \cup \{P\}$ -structures of the form $(M, P(M))$, with $P(M) \preceq_{\mathcal{L}} M \models T$ and satisfying certain genericity conditions.
- ▶ Common generalisation of Poizat's *belles paires* in a stable theory and Vassiliev's *generic pairs* in a SU-rank 1 theory.
- ▶ BPV show (among other things) that the following is equivalent:
 1. Loveliness is model-theoretically meaningful (i.e. saturated models of the theory of lovely pairs are lovely)
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Theorem

In the simple fusion context (as well as in other simple free amalgamation contexts), T_{ω} has the wncfp.