Generalised Hrushovski constructions

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Outline of the talk

1. Introduction
2. Generalised free fusion
3. A variant: bicoloured fields and bad fields
4. Generic automorphisms of Hrushovski constructions

The geometry of strongly minimal sets

- A **pregeometry** on a set X is given by a finitary closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the exchange lemma.
- Get notions of **dimension**, **independence**, **basis** etc.
- Ex: linear independence in a v.s.; alg. independence in a field.

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Trichotomy Conjecture (B. Zilber 1980)

Let T be a strongly minimal theory. Then, there are three cases:

- The geometry of T is **trivial**.
- T has a **locally modular** non-trivial geometry (projective or affine geometry over some skew-field).
- If T is not locally modular, T **interprets an algebraically closed field**.

Hrushovski amalgamation

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- most noteworthy: the **fusion construction** by Hrushovski, showing e.g. that there is a strongly minimal structure $(M, +_1, \cdot_1, +_2, \cdot_2)$ such that, for $i = 1, 2$, $(M, +_i, \cdot_i) \models ACF_{p_i}$.

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Theorem (E.Hrushovski 1992)

Let T_1 et T_2 be strongly minimal theories in (countable) disjoint languages, with definable multiplicities (DMP). Then, there is a strongly minimal theory $T \supseteq T_1 \cup T_2$.

Strongly minimal fusion: two steps

free fusion (we are mainly interested in this part in our talk):

- *Predimension function* (on finite subsets of $\mathcal{L}_1 \cup \mathcal{L}_2$-structures)

$$\delta(A) := \dim_1(A) + \dim_2(A) - |A|;$$

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  for $A \subseteq B$, $A \leq B$ if $\delta(A') \geq \delta(A)$ for all $A \subseteq A' \subseteq B$;
- put $\mathcal{C} := \{A \text{ finite } \mid \emptyset \leq A\}$;
- $(\mathcal{C}, \leq)$ is countable, has (AP), (JEP) and (HP);
- the Fraïssé limit of $(\mathcal{C}, \leq)$ is ω-stable of rank ω.

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collapse:
Amalgamate in a restricted class, uniformly bounding the number of solutions of sets of dim. 0 $\Rightarrow$ get a strongly minimal theory.

Question (Hrushovski 1992)

Let T_1 and T_2 be s.m. (in countable languages with DMP) which intersect in an infinite vector space over a finite field. Is it possible to find a s.m. completion of $T_1 \cup T_2$?

Fusion over sublanguages

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More generally: Is it possible to find a s.m. fusion T of two s.m. theories T_1, T_2 intersecting in some third theory T_0?

$$\begin{tikzcd}[column sep=small]
 & \exists T? \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
T_1 \arrow[dr, dash] & & T_2 \arrow[dl, dash] \\
 & T_0 &
\end{tikzcd}$$

A trivial example

- For G a group let $T_G =$ **theory of an infinite free G-action**. ($\Rightarrow$ T_G trivial strongly minimal)
- $G_0 \leq G_1, G_2$, $G := G_1 *_{G_0} G_2$
  $\Rightarrow$ T_G strongly minimal fusion of T_{G_1} and T_{G_2} over T_{G_0}.

$$\begin{tikzcd} & G & \\ G_1 \arrow[ur, hook] & & G_2 \arrow[ul, hook'] \\ & G_0 \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & T_G \arrow[dl, dash] \arrow[dr, dash] & \\ T_{G_1} & & T_{G_2} \\ & T_{G_0} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

A modular non-trivial example

- Let T_F be the **theory of an infinite vector space over** F, where F is a skew-field. ($\Rightarrow T_F$ modular strongly minimal)
- For $F_0 \subseteq F_1, F_2$, the ring $F_1 *_{F_0} F_2$ allows a field of fractions F $\Rightarrow T_F$ strongly minimal fusion of T_{F_1} and T_{F_2} over T_{F_0}.

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Two non-examples, two obstructions

1. Consider the following relative fusion context:
   - $T_0 = \mathbb{Q}$-vector spaces, with c, d (linearly independent) named,
   - $T_1 = \mathbb{Q}[i]$-vector spaces, with $i \cdot c = d$,
   - $T_2 = \mathbb{Q}(X)$-vector spaces, with $X \cdot c = d$.

   In every $M \models T_1 \cup T_2$, $\ker(i - X)$ defines a proper non-trivial $\mathbb{Q}$-subspace of M. In particular, $\mathrm{Th}(M)$ **is not of rank 1**.

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2. There is an example with the following properties:
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$\Rightarrow$ obstructions to a s.m. fusion, a **logical** and a **geometrical** one:

- Definability problems if T_0 is not ω-categorical.
- The geometrical interaction of the two structures can be *wild*.

Weaker requirements on the fusion theory

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- as the forking pregeometry attached to a regular type, where T is stable (ex.: free fusion in the original context);
- as before, with "simple" instead of "stable";
- as a pregeometry coming from some independence relation.

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- Examples of geometric theories:
  - Strongly minimal theories, more generally simple theories of SU-rank 1 (e.g. the random graph, pseudofinite fields).
  - RCF, more generally o-minimal theories.
  - $Th(\mathbb{Q}_p)$, as well as $ACVF$.
  - Reducts of (pre-)geometric theories are (pre-)geometric.

A context for the construction

Assumptions:
- T_1, T_2 are pregeometric theories;
- T_0 is strongly minimal and modular.

- Work with the predimension $\delta = \dim_1 + \dim_2 - \dim_0$.
- Obtain a *fusion class* $(\mathcal{C}, \leq)$. Structures in $\mathcal{C}$ are finitely $\langle \cdot \rangle$-generated ($\langle \cdot \rangle$ = transitive closure of acl_1 and acl_2).

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- $(\mathcal{C}, \leq)$ has (AP) $\Rightarrow$ rich structures do exist. Let T_ω be the theory of all rich structures.
- In general, $(\mathcal{C}, \leq)$ does not have (JEP) $\Rightarrow$ T_ω incomplete.
- saturated models of T_ω are not necessarily rich.

A context for axiomatisability

Definability assumptions:

- T_1, T_2 are geometric (pregeometric with elimination of $\exists^\infty$)
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Theorem

In this context, T_ω can be axiomatised. We obtain:

1. *Sufficiently saturated models of T_ω are rich.*
2. *In T_ω, every formula is equivalent to a boolean combinations of bounded existential formulas (assuming the T_i have QE in $\mathcal{L}_i$).*
3. *The completions of T_ω are determined by $\mathrm{qftp}_{\mathcal{L}}(\langle\emptyset\rangle)$ (here, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$).*

Simple fusion

Context:
- T_1, T_2 simple SU-rank 1 (in particular geometric)
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- The proof uses the Theorem of Kim-Pillay. Difficult to establish: the *Independence Theorem*.
- Note the similarities with results of Chatzidakis-Pillay about stable theories with a generic automorphism (I will come to that later).

Fact

In the following cases, $T_1 \supseteq T_0$ satisfies condition **A***:*

1. T_0 *with trivial pregeometry,* T_1 *arbitrary*
2. T_1 *strongly minimal,* T_0 *arbitrary.*
3. F *a pseudofinite field,* $T_1 = Th(F, +, \times)$, $T_0 = Th(F, +)$

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2. *T_1 strongly minimal, T_0 arbitrary.*
3. *F a pseudofinite field, $T_1 = Th(F, +, \times)$, $T_0 = Th(F, +)$*

Corollary

1. *Two arbitrary* SU*-rank 1 theories can be fused into a simple theory of* SU*-rank $\leq \omega$.*
2. *For ω-categorical T_0, and s.m. expansions $T_1, T_2 \supseteq T_0$, there is a simple fusion of T_1 and T_2 over T_0 (of* SU*-rank $\leq \omega$).*
3. *There is a simple structure $(F, +, \times_1, \times_2)$ of* SU*-rank ω such that $(F, +, \times_1) \models PSF_p$ and $(F, +, \times_2) \models ACF_p$ (for $p > 0$).*

ω-stable free fusion

Setting:
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Theorem (Hasson, H. 2006)

In the above setting, T_ω is complete ω-stable with a unique generic type of rank ω.
A detailed description of the types to be collapsed can be given.

Strongly minimal fusion: collapse results

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- Not hard to see: the arbitrary case reduces to one of the previous cases.

A variant: Poizat's bicoloured fields I

Poizat (1999,2001): Construction of various expansions of algebraically closed fields K (adding a new predicate).

Black fields:

- $N^K \subseteq K$ distinguished *subset*, $char(K)$ arbitrary (but fixed).
- Predimension $\delta((K, N^K)) = 2\,\mathrm{tr.deg}(K) - |N^K|$
- Free amalgamation $\Rightarrow$ black field of Morley rank $\omega \cdot 2$.
- Collapse (Poizat, Baldwin-Holland) to a black field of MR 2.

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Red fields:

- $R^K \subseteq K$ distinguished *additive subgroup*, $char(K) = p > 0$.
- Predimension $\delta((K, R^K)) = 2\,\mathrm{tr.deg}(K) - \mathrm{l.dim}_{\mathbb{F}_p}(R^K)$
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Green fields:

- $\ddot{\mathrm{U}}^K \subseteq K^*$ is a distinguished *multiplicative subgroup*, with $\ddot{\mathrm{U}}$ divisible and torsion free, $char(K) = 0$.
- Predimension $\delta((K, \ddot{\mathrm{U}}^K)) = 2\,\mathrm{tr.deg}(K) - \mathrm{l.dim}_{\mathbb{Q}}(\ddot{\mathrm{U}}^K)$
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Remark

Bicoloured structures have simple analogues, e.g. for $p > 0$, there is $F \models Psf_p$ with an additive subgroup $R^F \leq F$ such that (F, R) is supersimple of SU*-rank $\omega \cdot 2$.*

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Algebraicity Conjecture (Cherlin-Zilber)

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  Bad Fields: structures $(K, \ddot{\mathrm{U}})$ of finite Morley rank with $K \models ACF$ and $\ddot{\mathrm{U}}$ a proper infinite subgroup of $(K^*, \cdot)$.
- Longstanding open question of B. Zilber: Do bad fields exist?

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Theorem (Baudisch, Martin Pizarro, H., Wagner)

There is a bad field $(K, \ddot{U})$ in char. 0, obtained by collapsing Poizat's green field ("bad field of infinite rank").

Generic automorphisms of stable theories

- Let T be a stable, complete and model-complete $\mathcal{L}$-theory. (in case T is not model-complete, we morleyise first)
- $(M, \sigma) \models T_\sigma$ iff $M \models T$ and $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M)$ (in $\mathcal{L} \cup \{\sigma\}$)
- We say the *generic automorphism is axiomatisable in T* $:\Leftrightarrow$ T_σ admits a model-companion (denoted TA).

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Fact (Chatzidakis-Pillay)

TA is a simple theory. Every formula is equivalent to a boolean combination of bounded existential formulas (assuming T has QE), and its completions are given by the action of σ on $\mathrm{acl}(\emptyset)$

Theorem

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2. *Axiomatisability also holds in various other theories obtained by a (free) Hrushovski amalgamation, e.g.*
    - *Hrushovski's ab initio construction;*
    - *black fields and red fields;*
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Fact (Hasson-Hrushovski)

For strongly minimal T, the generic automorphism is axiomatisable iff T has the DMP.

Idea of the proof.

We first establish a general criterion: The generic automorphism is axiomatisable if we have a *notion of genericity* s.t.

- there are "enough" formulas containing a single generic type;
- "containing a single generic type" is definable in families;
- "projecting the generic on the generic" is definable in families.

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Remark

In the corresponding collapsed versions, the axiomatisability of the generic automorphism follows from results of Chatzidakis-Pillay. It can also be shown using the above general criterion.

Compatibility with other generic constructions: lovely pairs

- Ben-Yaacov, Pillay and Vassiliev introduced **Lovely pairs** (of models) of a simple theory T, i.e. $\mathcal{L} \cup \{P\}$-structures of the form $(M, P(M))$, with $P(M) \preccurlyeq_{\mathcal{L}} M \models T$ and satisfying certain genericity conditions.
- Common generalisation of Poizat's *belles paires* in a stable theory and Vassiliev's *generic pairs* in a SU-rank 1 theory.
- BPV show (among other things) that the following is equivalent:
  1. Lovelyness is model-theoretically meaningful (i.e. saturated models of the theory of lovely pairs are lovely)
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Theorem

In the simple fusion context (as well as in other simple free amalgamation contexts), T_ω has the wnfcp.