Paul Isaac Bernays (1888-1977)

1912 Doctorate at Göttingen with Landau

1913 Habilitation at Zürich

–1917 Assistant to Zermelo

1917– Assistant to Hilbert at Göttingen

1919 Second Habilitation

1922 Professor Extraordinarius

1934– Eidgenössische Technische Hochschule (ETH) at Zürich

Bernays to Gödel, January 1931:

I have laid out a modified version of von Neumann's set theory which, first of all, establishes a closer relation to the ordinary logical processes of set formation, and furthermore eliminates various unnecessary deviations from Zermelo's system and makes the formulation of the axioms more easily understandable.

Gödel, January 1931:

In case we adopt a type-free construction of mathematics, as is done in the axiom system of set theory, axioms of cardinality (that is, axiom postulating the existence of sets of ever higher cardinality) take the place of type extensions, and it follows that certain arithmetic propositions that are undecidable in Z [first-order Peano arithmetic] become decidable by axioms of cardinality, for example, by the axiom that there exist sets whose cardinality is greater than every α_n, where $\alpha_0 = \aleph_0$, $\alpha_{n+1} = 2^{\alpha_n}$.

The Axiomatization

I. Axioms of Extensionality

(1) $\forall x(x \in a \longleftrightarrow x \in b) \longrightarrow a = b.$

(2) $\forall x(x \,\eta\, A \longleftrightarrow x \,\eta\, B) \longrightarrow A = B.$

II. Axioms of Direct Construction of Sets

(1) $\exists a \forall x(x \notin a).$

(2) $\forall a \forall b \exists c \forall x(x \in c \longleftrightarrow x \in a \vee x = b).$

III. Axioms for Construction of Classes

a(1) $\forall a \exists A \forall x (x \,\eta\, A \longleftrightarrow x = a)$.

a(2) $\forall A \exists B \forall x (x \,\eta\, B \longleftrightarrow \neg x \,\eta\, A)$.

a(3) $\forall A \forall B \exists C$
$\forall x (x \,\eta\, C \leftrightarrow x \,\eta\, A \;\&\; x \,\eta\, B)$.

b(1) $\exists A \forall x$
$(x \,\eta\, A \longleftrightarrow \exists a \forall y (y \in x \longleftrightarrow y = a))$.

b(2) $\exists A \forall x$
$(x \,\eta\, A \longleftrightarrow \exists a \exists b (a \in b \;\&\; x = \langle a, b \rangle))$.

b(3) $\forall A \exists B \forall x$
$(x \,\eta\, B \longleftrightarrow \exists a \exists b (x = \langle a, b \rangle \;\&\; a \,\eta\, A))$.

c(1) $\forall A \exists B \forall x (x \,\eta\, B \longleftrightarrow \exists y (\langle x, y \rangle \,\eta\, A))$

c(2) $\forall A \exists B \forall a \forall b$
$(\langle a, b \rangle \in B \longleftrightarrow \langle b, a \rangle \in A)$.

c(3) $\forall A \exists B \forall a \forall b \forall c$
$(\langle \langle a, b \rangle, c \rangle \in B \longrightarrow \langle a, \langle b, c \rangle \rangle \in A)$.

IV. Axiom of Choice

Every relation C has a subclass which is a function and has the same domain.

V. Axioms Concerning the Representation of Classes by Sets

a. (Separation)
$\forall a \forall A \exists b \forall x (x \in b \longleftrightarrow x \in a \ \& \ x \, \eta \, A)$.
b. (Replacement)
If the domain of a one-to-one correspondence is represented by a set, then so is the range.
c. (Union)
$\forall a \exists b \forall x (x \in b \longleftrightarrow \exists y (y \in a \ \& \ x \in y))$.
d. (Power Set)
$\forall a \exists b \forall x (x \in b \longleftrightarrow x \subseteq a)$.

VI. Axiom of Infinity
There is a set in one-to-one correspondence with a proper subclass.

VII. Axiom of Foundation

$$\forall A(\exists x\,\eta\,A \longrightarrow \\ \exists b(b\,\eta\,A \;\&\; \neg\exists z(z \in b \;\&\; z\,\eta\,A))).$$

von Neumann's axiom IV 2:

A class A is not (represented by) a set exactly when there is a surjection of A onto V.

von Neumann, 1925:

Axiom IV 2, finally, deviates quite essentially from what Zermelo and Fraenkel have, and indeed it is the distinctive feature of our axiomatization. It is, to be sure, related in a certain sense to the axioms of separation and replacement, but it goes much further … IV 2 occupies an altogether central position in the axiom system; it several cases it enables us to prove that a set is "not too big", and finally it yields the well-ordering theorem.

The Cumulative Hierarchy:

$$V_0 = \emptyset;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ \text{and}$$

$$V_\delta = \bigcup_{\alpha<\delta} V_\alpha \text{ for limit ordinals } \delta\,.$$

With Foundation,
von Neumann's axiom IV 2 $\longleftrightarrow$
Choice IV & Replacement Vb:

If A is (not represented by) a set, then A is the union of the fibers $A \cap (V_{\alpha+1} - V_\alpha)$ by Foundation which are non-empty for arbitrarily large α by Replacement. But each such fiber can be well-ordered and these well-orderings can be put together, all by Choice, to get a bijection between A and the class On of all ordinals.

Gödel's axioms of inversion:

(B7) $\forall A \exists B \forall x \forall y \forall z$
$(\langle x, \langle y, z\rangle\rangle \in B \longleftrightarrow \langle y, \langle z, x\rangle\rangle \in A)\,.$

(B8) $\forall A \exists B \forall x \forall y \forall z$
$(\langle x, \langle y, z\rangle\rangle \in B \longleftrightarrow \langle x, \langle z, y\rangle\rangle \in A)\,.$

Bernays' *Axiomatic Set Theory* 1958:

Class variables but not quantified.

Class terms $\{x \mid \varphi(x)\}$.

The conversion scheme

$$\varphi(a) \longleftrightarrow a \in \{x \mid \varphi(x)\}.$$

A 1	Emptyset
A 2	$a \cup \{b\}$
A 3	$\bigcup_{x \in a} t(x)$ for operators t
A 4	Power set
A 5	Choice (for sets)
A 6	Infinity
A 7	Foundation (for sets)

Reflection Principles

ZF Reflection Principle:

For any ZF formula $\varphi(v_1, \ldots, v_n)$ and any ordinal β, there is a limit $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \quad \textit{iff} \quad \varphi^{V_\alpha}[x_1, \ldots, x_n].$$

Bernays' Restricted Schema:

$$\varphi \rightarrow \exists y(\mathrm{Trans}(y) \ \& \ \varphi^y).$$

Bernays' Class Reflection Scheme:

$$\begin{aligned}
&\varphi(a_1, \ldots, a_n, A_1, \ldots, A_r) \longrightarrow \\
&\qquad \exists y(\mathrm{Trans}(y) \ \& \\
&\qquad \varphi^y(a_1, \ldots, a_n, A_1 \cap y, \ldots, A_r \cap y)).
\end{aligned}$$