The Σ_1 collection principle over finite fragments of bounded arithmetic

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Bounded arithmetic

- Language L_2 contains 0, 1, $\leq$, $+$, $\times$, $\#$, $|\cdot|$, $\lfloor\frac{\cdot}{2}\rfloor$.
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  $|x|$ is $\lceil \log(x+1) \rceil$, $x\#y$ is $2^{|x|\cdot|y|}$.
- Quantifiers bounded by $|t|$ called sharply bounded.
- Hierarchy of bounded formulae: the class $\hat{\Sigma}^b_n$ contains formulae of the form:

$$\exists y_1 < t_1 \, \forall y_2 < t_2 \ldots Q y_n < t_n \, \psi$$

  with sharply bounded ψ. $\hat{\Pi}^b_n$ is dual to $\hat{\Sigma}^b_n$.

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- S^n_2 is like T^n_2 but with conclusion of induction restricted to lengths (i.e. elements in the range of $|\cdot|$).
- $S^1_2 \subseteq T^1_1 \subseteq S^2_2 \subseteq T^2_2 \subseteq S^3_2 \ldots$

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These theories are studied for a number of reasons, e.g. connections to computational complexity theory.
Major question: is there n such that $S_2 = S_2^n$.

Collection

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- $B\Sigma_1$ is unprovable in S_2 (Paris-Kirby), but all known proofs require exponentiation. A long-standing open problem: does $S_2 + \neg\exp$ prove $B\Sigma_1$?
- $S_2^n + \neg\exp$ does not prove $B\Sigma_1$ (folklore), but the complexity of known unprovable instances grows with n.
- $T + B\Sigma_1$ is Π_2-conservative over T for any reasonably strong $T \subseteq S_2$ (Buss).

Main result

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Theorem

Let $n \geq 1$. If $S^n_2 \nvdash WPHP(\hat{\Sigma}^b_n)$, then $B\Sigma_1$ is not finitely axiomatizable over S^n_2. The same result holds with T^n_2 in place of S^n_2.

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Remark: some kind of assumption is needed for the theorem, since if $S^n_2 = S_2$, then $B\Sigma_1$ is finitely axiomatizable over S^n_2. There are some results suggesting that our assumption about *WPHP* is plausible.

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- Build a cofinal $\hat{\Sigma}^b_{k+2}$-elementary extension of M to a $\hat{\Sigma}^b_{k+3}$-maximal model (i.e., M_+ such that $M_+ \preceq_{\hat{\Sigma}^b_{k+2}} K \models S^n_2$ implies $M_+ \preceq_{\hat{\Sigma}^b_{k+3}} K$).

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- This can be done so that M_+ satisfies $B\exists\hat{\Pi}^b_k$.
- Use the fact that $WPHP$ still fails at a in M_+ to show that M_+ cannot satisfy $B\exists\hat{\Pi}^b_{k+2}$.

Extending M to a maximal model

The construction of a $\hat{\Sigma}^b_{k+3}$-maximal extension M_+ of M is standard: build a chain

$$M \preceq_{\hat{\Sigma}^b_{k+2}} M_1 \preceq_{\hat{\Sigma}^b_{k+2}} M_2 \preceq_{\hat{\Sigma}^b_{k+2}} \dots$$

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Preserving cofinality and $B\exists\hat{\Pi}^b_k$ is not difficult: by compactness and some standard techniques from models of arithmetic, we can actually guarantee that each M_s satisfies all of $B\Sigma_1$.
$B\exists\hat{\Pi}^b_k$ is the amount that gets preserved in the union.

Why M_+ does not satisfy collection

In M_+, any $\hat{\Sigma}^b_{k+3}$ formula $\psi(x)$ is equivalent to "$\psi(x)$ is consistent with S^n_2 plus the $\hat{\Pi}^b_{k+2}$ diagram".
Observation (Adamowicz, K. 2007): this consistency statement can be expressed as a $\forall\hat{\Sigma}^b_{k+2}$ formula.

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If M_+ satisfies $B\exists\hat{\Pi}^b_{k+2}$, then on each segment $[0, d]$, the witness for "$\psi(x)$ or there is an inconsistency proof from $\psi(x)$" can be bounded. So, each $\hat{\Sigma}^b_{k+3}$ formula becomes equivalent on $[0, d]$ to a $\hat{\Pi}^b_{k+3}$ formula .

Why M_+ does not satisfy collection (cont'd)

However (K., Thapen 2008): In a model of $S^n_2 + \neg WPHP(\hat{\Sigma}^b_n)$, it cannot happen that each $\hat{\Sigma}^b_m$ formula is equivalent to a $\hat{\Pi}^b_m$ formula ("the polynomial hierarchy does not collapse").
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By analyzing the argument, one may see that it excludes even equivalence of $\hat{\Sigma}^b_m$ and $\hat{\Pi}^b_m$ formulae on a sufficiently large segment $[0, d]$. Thus, M_+ cannot satisfy $B\exists\hat{\Pi}^b_{k+2}$. $\Box$