Automorphisms of models of PA: The neglected cases.

Roman Kossak
City University of New York

July 3, 2008

In memory of Henryk Kotlarski 1949-2008

OVERVIEW

I Models of PA in the 1980s

II Automorphisms of models of PA

III Automorphism groups of models of PA

IV The neglected cases. What is left to be done?

MODEL THEORY OF PA IN 1980s

- Hájek, Kirby, McAloon, Paris, Pudlák: Combinatorial properties of cuts, indicators, independence results.

MODEL THEORY OF PA IN 1980s

- Hájek, Kirby, McAloon, Paris, Pudlák: Combinatorial properties of cuts, indicators, independence results.
- Barwise, Kaufmann, Kaye, Lessan, Macintyre, Ressayre, Schlipf, Smoryński, Wilmers: Model theory of recursively saturated structures. Resplendence. Expandability. Automorphisms.

MODEL THEORY OF PA IN 1980s

- Hájek, Kirby, McAloon, Paris, Pudlák: Combinatorial properties of cuts, indicators, independence results.
- Barwise, Kaufmann, Kaye, Lessan, Macintyre, Ressayre, Schlipf, Smoryński, Wilmers: Model theory of recursively saturated structures. Resplendence. Expandability. Automorphisms.
- Kotlarski, Krajewski, Lachlan, Murawski, Smith: Nonstandard satisfaction classes.

MODEL THEORY OF PA IN 1980s

- Hájek, Kirby, McAloon, Paris, Pudlák: Combinatorial properties of cuts, indicators, independence results.
- Barwise, Kaufmann, Kaye, Lessan, Macintyre, Ressayre, Schlipf, Smoryński, Wilmers: Model theory of recursively saturated structures. Resplendence. Expandability. Automorphisms.
- Kotlarski, Krajewski, Lachlan, Murawski, Smith: Nonstandard satisfaction classes.
- Abramson, Blass, Gaifman, Harrington, Mills, Schmerl, Wilkie: Extensions. Definable types. Iterated ultrapowers. Lattices of elementary submodels. Uncountable models with interesting second-order properties.

MODEL THEORY OF PA IN 1980s

- Hájek, Kirby, McAloon, Paris, Pudlák: Combinatorial properties of cuts, indicators, independence results.
- Barwise, Kaufmann, Kaye, Lessan, Macintyre, Ressayre, Schlipf, Smoryński, Wilmers: Model theory of recursively saturated structures. Resplendence. Expandability. Automorphisms.
- Kotlarski, Krajewski, Lachlan, Murawski, Smith: Nonstandard satisfaction classes.
- Abramson, Blass, Gaifman, Harrington, Mills, Schmerl, Wilkie: Extensions. Definable types. Iterated ultrapowers. Lattices of elementary submodels. Uncountable models with interesting second-order properties.
- Harnik, Knight, Marker, Nadel, Pabion, Richard, Solovay: Turing degrees. Scott sets. Presburger Arithmetic. Reducts. Other...

Henryk Kotlarski, Automorphisms of Countable Recursively Saturated Models of PA: a Survey, Notre Dame Journal of Formal Logic Volume 36, Number 4, 505-518 (1995)

Henryk Kotlarski, Automorphisms of Countable Recursively Saturated Models of PA: a Survey, Notre Dame Journal of Formal Logic Volume 36, Number 4, 505-518 (1995)

> *Let me begin with the pre-history of the subject. The question whether PA has a model with a nontrivial automorphism (i.e., such that its elements cannot be individualized) was due to Hasenjäger. It was solved positively by Ehrenfeucht and Mostowski [3]. Their result and the idea of indiscernibility is nowadays so well known that Hodges [5] writes "today model theorists use it at least once a week," so let me omit the statement of the Ehrenfeucht-Mostowski Theorem.*

THREE PAPERS ON EXTENDING AUTOMORPHISMS

1. *Results on automorphisms of recursively saturated models of PA*, Fund. Math. 129 (1), 9-15 (1988).
2. *On extending of automorphisms of models of Peano arithmetic*, Fund. Math. 149 (3), 245-263 (1996).
3. *More on extending automorphisms of models of Peano arithmetic*, to appear in Fund. Math.

AUTOMORPHISMS OF COUNTABLE RECURSIVELY SATURATED STRUCTURES

Let $\mathfrak{M}$ be countable and recursively saturated.

Proposition

For every infinite $A \in \mathrm{Def}(\mathfrak{M})$ *there are* $a, b \in A$ *such that* $a \neq b$ *and* $\mathrm{tp}(a) = \mathrm{tp}(b)$*.*

AUTOMORPHISMS OF COUNTABLE RECURSIVELY SATURATED STRUCTURES

Let $\mathfrak{M}$ be countable and recursively saturated.

Proposition

For every infinite $A \in \mathrm{Def}(\mathfrak{M})$ *there are* $a, b \in A$ *such that* $a \neq b$ *and* $\mathrm{tp}(a) = \mathrm{tp}(b)$.

Proposition

For all $a, b \in \mathfrak{M}$, *if* $\mathrm{tp}(a) = \mathrm{tp}(b)$, *then there is* $f \in \mathrm{Aut}(\mathfrak{M})$ *such that* $f(a) = b$.

AUTOMORPHISMS OF COUNTABLE RECURSIVELY SATURATED STRUCTURES

Let $\mathfrak{M}$ be countable and recursively saturated.

Proposition

For every infinite $A \in \mathrm{Def}(\mathfrak{M})$ *there are* $a, b \in A$ *such that* $a \neq b$ *and* $\mathrm{tp}(a) = \mathrm{tp}(b)$.

Proposition

For all $a, b \in \mathfrak{M}$, *if* $\mathrm{tp}(a) = \mathrm{tp}(b)$, *then there is* $f \in \mathrm{Aut}(\mathfrak{M})$ *such that* $f(a) = b$.

Proposition

$|\mathrm{Aut}(\mathfrak{M})| = 2^{\aleph_0}$.

WHAT IS SPECIAL ABOUT PA?
EHREFEUCHT-GAIFMAN LEMMA

Theorem

If $M \models \mathrm{PA}$, $f \in \mathrm{Aut}(M)$, *and* $f(a) \neq a$, *then* $f(a) \notin \mathrm{Scl}(a)$[1].

[1] $\mathrm{Scl}(a)$ is the definable closure of a in M.

AUTOMORPHIC IMAGES

Definition

For $X \subseteq \mathfrak{M}$, $\mathfrak{a}(X) = |\{f(X) : f \in \mathrm{Aut}(\mathfrak{M})\}|$

Basic facts:

AUTOMORPHIC IMAGES

Definition

For $X \subseteq \mathfrak{M}$, $\mathfrak{a}(X) = |\{f(X) : f \in \mathrm{Aut}(\mathfrak{M})\}|$

Basic facts:

$$X \in \mathrm{Def}_0(\mathfrak{M}) \Rightarrow \mathfrak{a}(X) = 1$$

AUTOMORPHIC IMAGES

Definition

For $X \subseteq \mathfrak{M}$, $\mathfrak{a}(X) = |\{f(X) : f \in \mathrm{Aut}(\mathfrak{M})\}|$

Basic facts:

$$X \in \mathrm{Def}_0(\mathfrak{M}) \Rightarrow \mathfrak{a}(X) = 1$$

$$X \in \mathrm{Def}(\mathfrak{M}) \Rightarrow \mathfrak{a}(X) \leq \aleph_0$$

AUTOMORPHIC IMAGES

Definition

For $X \subseteq \mathfrak{M}$, $\mathfrak{a}(X) = |\{f(X) : f \in \mathrm{Aut}(\mathfrak{M})\}|$

Basic facts:

$$X \in \mathrm{Def}_0(\mathfrak{M}) \Rightarrow \mathfrak{a}(X) = 1$$

$$X \in \mathrm{Def}(\mathfrak{M}) \Rightarrow \mathfrak{a}(X) \leq \aleph_0$$

$$\mathfrak{a}(X) > \aleph_0 \Rightarrow \mathfrak{a}(X) = 2^{\aleph_0} \text{ (Kueker-Reyes)}$$

$\mathfrak{a}(X)$ IN MODELS OF PA

Assumption from now on: $M \models \mathrm{PA}$ is countable and recursively saturated

$\mathfrak{a}(X)$ IN MODELS OF PA

Assumption from now on: $M \models$ PA is countable and recursively saturated

Theorem (Schmerl)

a. $\forall X \subseteq M$, $\mathfrak{a}(X) \in \{1, \aleph_0, 2^{\aleph_0}\}$

$\mathfrak{a}(X)$ IN MODELS OF PA

Assumption from now on: $M \models$ PA is countable and recursively saturated

Theorem (Schmerl)

a. $\forall X \subseteq M$, $\mathfrak{a}(X) \in \{1, \aleph_0, 2^{\aleph_0}\}$
b. $\forall X \in \mathrm{Def}(M) \setminus \mathrm{Def}_0(M), \mathfrak{a}(X) = \aleph_0$.

MANY AUTOMORPHIC IMAGES

MANY AUTOMORPHIC IMAGES

Theorem (RK, Kotlarski)

$(M, X) \models \mathrm{PA}^* \text{ and } X \notin \mathrm{Def}(M) \Rightarrow \mathfrak{a}(X) = 2^{\aleph_0}$

MANY AUTOMORPHIC IMAGES

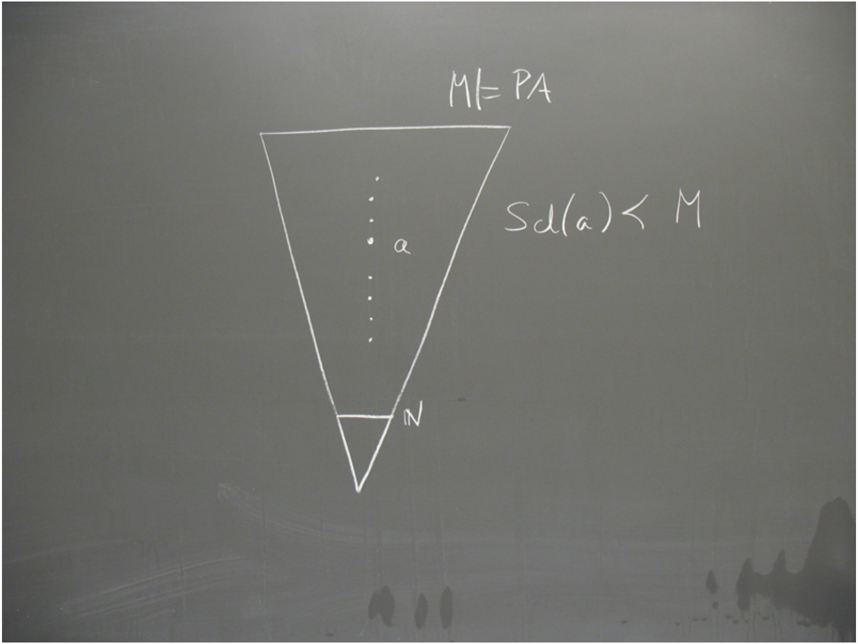
Theorem (RK, Kotlarski)

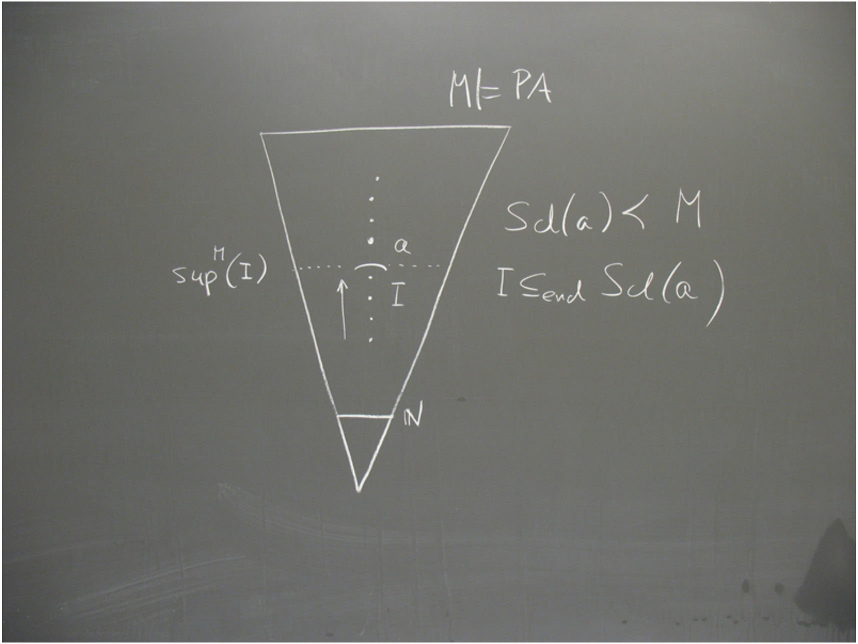
$(M, X) \models \mathrm{PA}^*$ and $X \notin \mathrm{Def}(M) \Rightarrow \mathfrak{a}(X) = 2^{\aleph_0}$

Theorem

Let $I \subseteq_{\mathrm{end}} M$. Then $\mathfrak{a}(I) < 2^{\aleph_0}$ iff $\exists a \in M$

$$I = \sup[\mathrm{Scl}(a) \cap I] \text{ or } I = \inf[\mathrm{Scl}(a) \cap (M \setminus I)]$$


M ⊨ PA
Sd(a) < M
a
ℕ


$M \models PA$
$Sd(a) < M$
a
$\sup^M(I)$
I
$I \subseteq_{end} Sd(a)$
$\mathbb{N}$

MANY AUTOMORPHIC IMAGES

From now on $M \models \mathrm{PA}$ is countable and recursively saturated

Theorem (RK, Kotlarski)

$(M, X) \models \mathrm{PA}^*$ and $X \notin \mathrm{Def}(M) \Rightarrow \mathfrak{a}(X) = 2^{\aleph_0}$

Theorem

Let $I \subseteq_{\mathrm{end}} M$. *Then* $\mathfrak{a}(I) < 2^{\aleph_0}$ *iff* $\exists a \in M$

$$I = \sup[\mathrm{Scl}(a) \cap I] \text{ or } I = \inf[\mathrm{Scl}(a) \cap (M \setminus I)]$$

MANY AUTOMORPHIC IMAGES

From now on $M \models \mathrm{PA}$ is countable and recursively saturated

Theorem (RK, Kotlarski)

$(M, X) \models \mathrm{PA}^*$ and $X \notin \mathrm{Def}(M) \Rightarrow \mathfrak{a}(X) = 2^{\aleph_0}$

Theorem

Let $I \subseteq_{\mathrm{end}} M$. Then $\mathfrak{a}(I) < 2^{\aleph_0}$ iff $\exists a \in M$

$$I = \sup[\mathrm{Scl}(a) \cap I] \text{ or } I = \inf[\mathrm{Scl}(a) \cap (M \setminus I)]$$

Theorem (RK, Kotlarski, independently Tzouvaras)

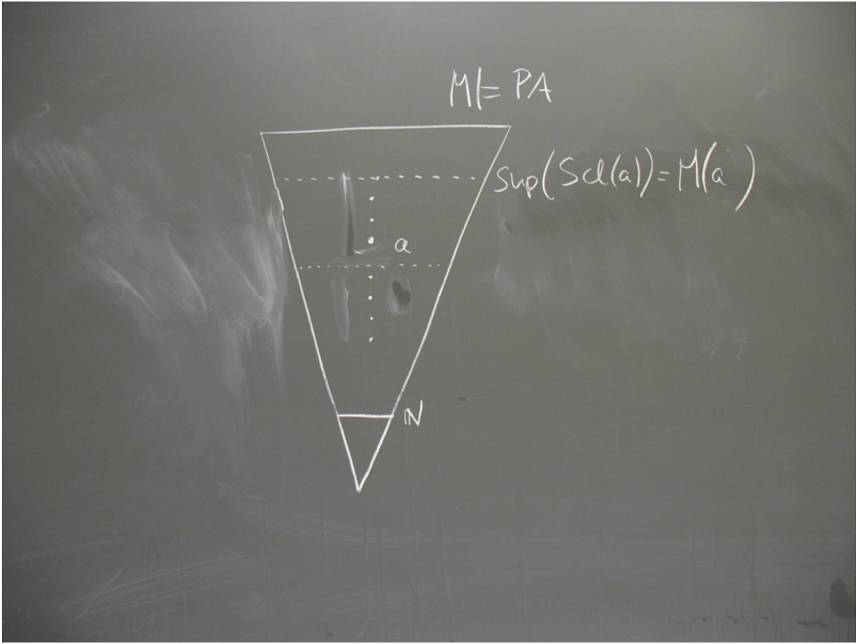
$f \in \mathrm{Aut}(M)$ and $f \neq id \Rightarrow |\{f^g : g \in \mathrm{Aut}(M)\}| = \mathfrak{a}(f) = 2^{\aleph_0}$

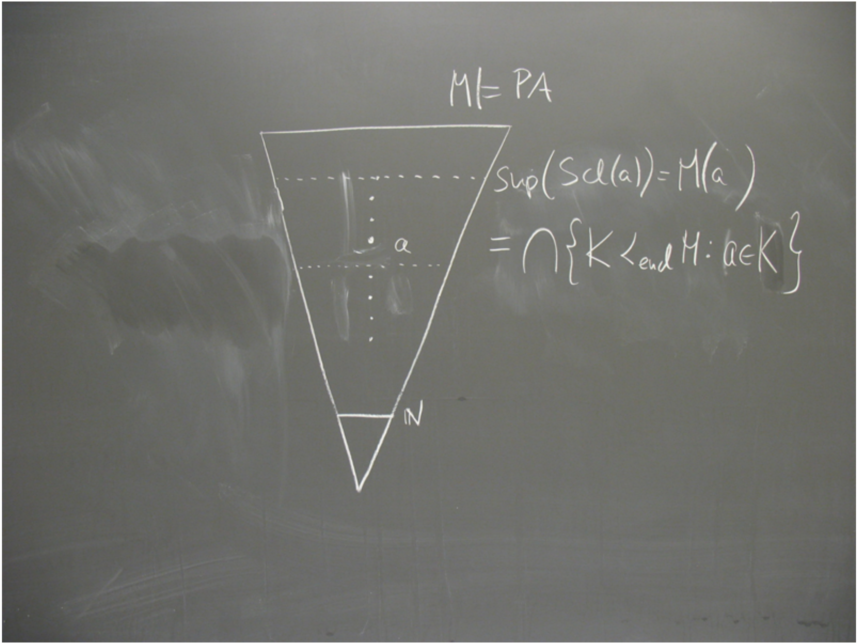
MOVING GAPS LEMMA

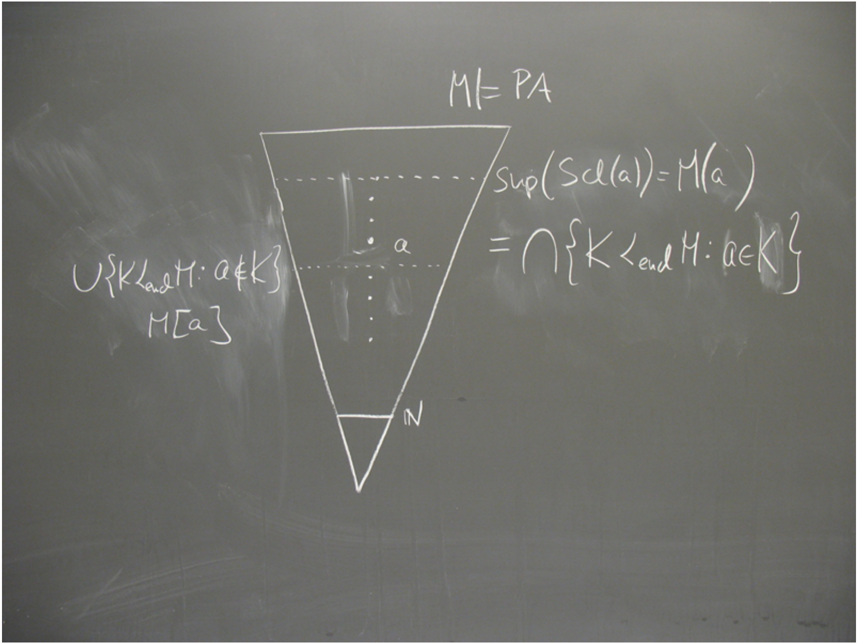
Theorem (Kotlarski)

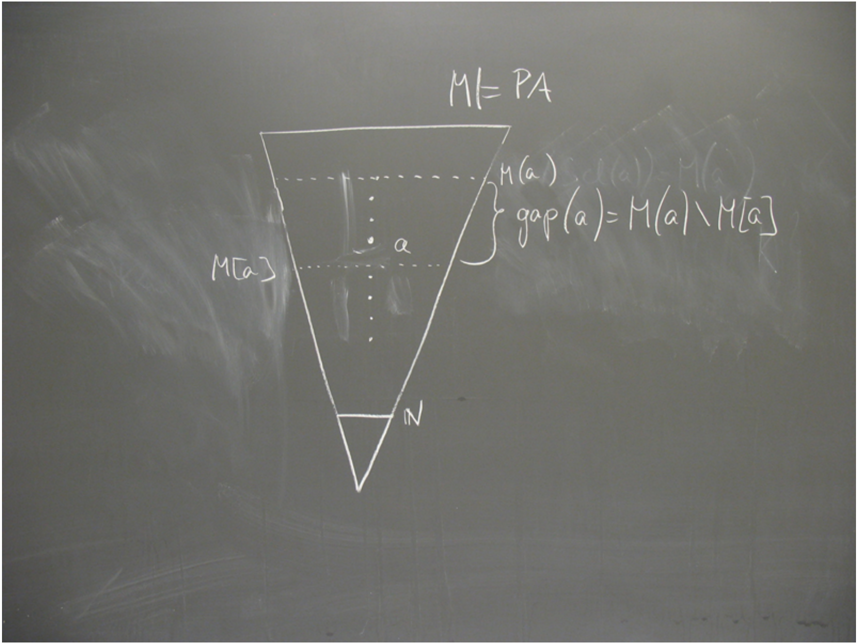
If $f \in \mathrm{Aut}(M) \setminus \{id\}$*, then for arbitrarily large* $a \in M$*,*

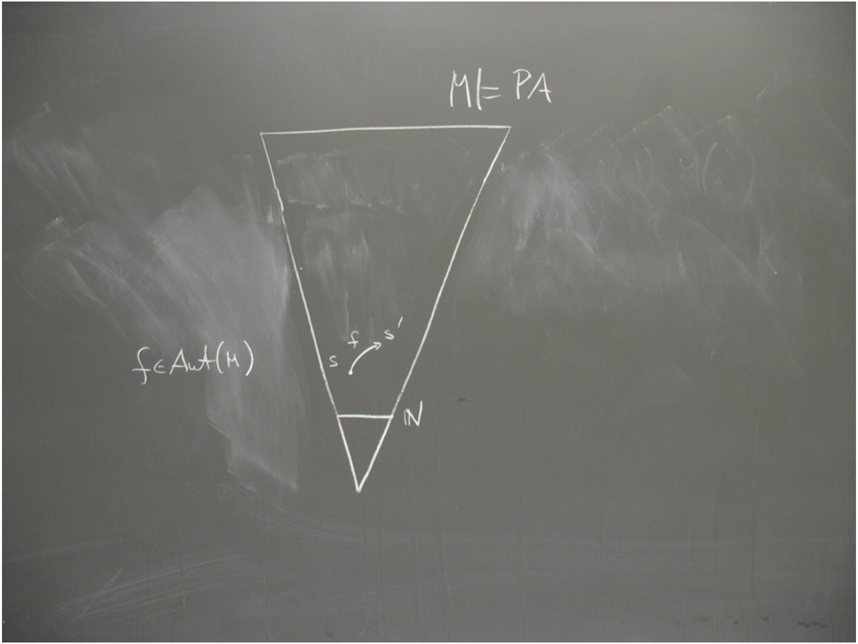
$$f(a) \notin \mathrm{gap}(a).$$

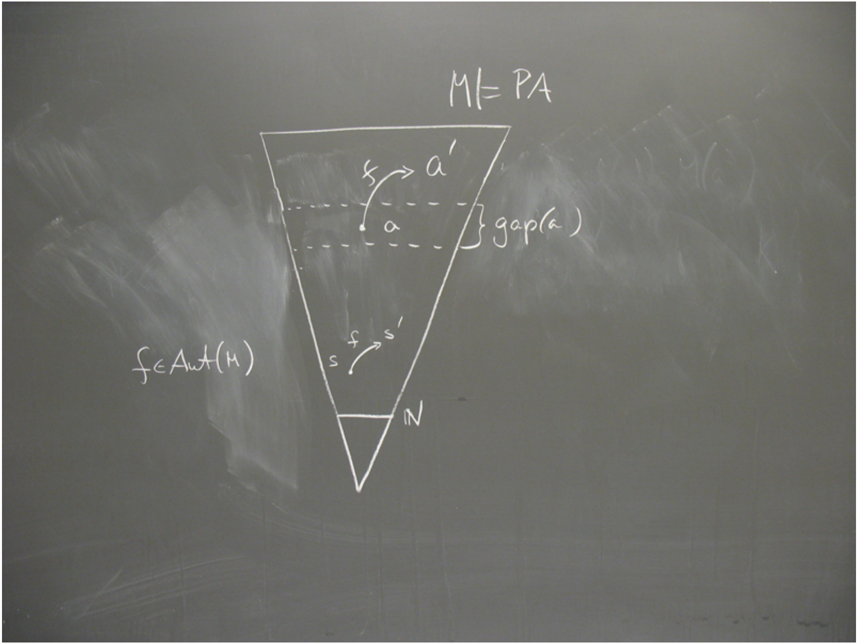

M ⊨ PA
Sup(Scl(a)) = M(a)
a
ℕ


$M \models PA$
$\sup(Scl(a)) = M(a)$
$= \bigcap\{K <_{end} M : a \in K\}$
a
$\mathbb{N}$


$M \models PA$
$\sup(\mathrm{Scl}(a)) = M(a)$
$= \bigcap\{K <_{end} M : a \in K\}$
a
$\bigcup\{K <_{end} M : a \notin K\}$
$M[a]$
$\mathbb{N}$


M ⊨ PA
M(a)
gap(a) = M(a) \ M[a]
a
M[a]
ℕ


M ⊨ PA
f ∈ Aut(M)
s
f
s′
ℕ


M ⊨ PA
f
a'
a
gap(a)
f ∈ Aut(M)
s
f
s'
ℕ

MOVING GAPS LEMMA

Theorem (Kotlarski)

If $f \in \mathrm{Aut}(M) \setminus \{id\}$, then for arbitrarily large $a \in M$,

$$f(a) \notin \mathrm{gap}(a).$$

MOVING GAPS LEMMA

Theorem (Kotlarski)

If $f \in \mathrm{Aut}(M) \setminus \{id\}$*, then for arbitrarily large* $a \in M$*,*

$$f(a) \notin \mathrm{gap}(a).$$

Corollary

If $p(v)$ *is an unbounded type realized in* M*,* $f \in \mathrm{Aut}(M) \setminus \{id\}$*, then there is* $a \in p^M$ *such that* $f(a) \neq a$*.*

MOVING GAPS LEMMA

Theorem (Kotlarski)

If $f \in \mathrm{Aut}(M) \setminus \{id\}$, then for arbitrarily large $a \in M$,

$$f(a) \notin \mathrm{gap}(a).$$

Corollary

If $p(v)$ is an unbounded type realized in M, $f \in \mathrm{Aut}(M) \setminus \{id\}$, then there is $a \in p^M$ such that $f(a) \neq a$.

Corollary

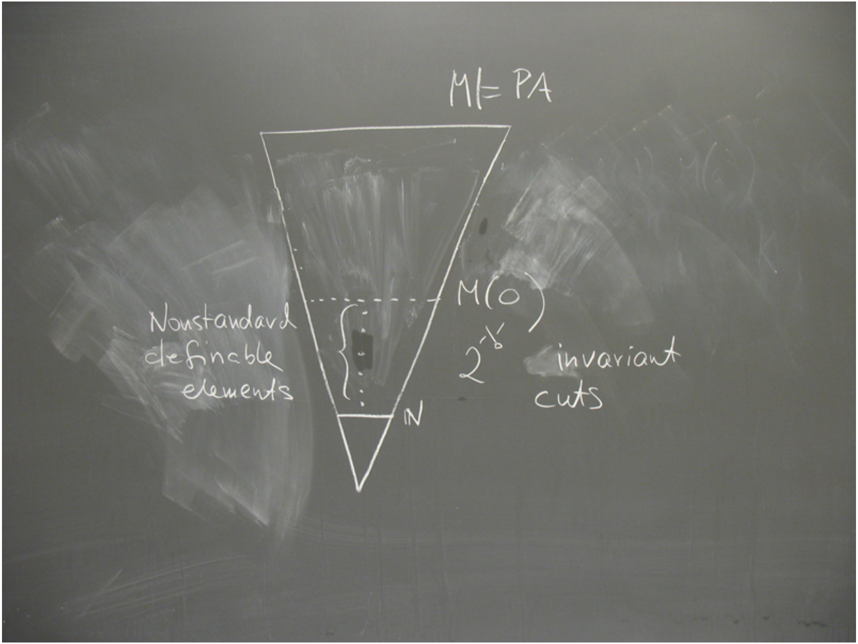
If $K \prec M$ and $K \cap \mathrm{gap}(a) \neq \emptyset$ for all $a \in M$, then $K = M$.

THE AUTOMORPHISM GROUP

$M \models$ PA countable and recursively saturated, $G = \mathrm{Aut}(M)$.

Theorem (Kaye)

If $H \lhd G$ is closed then $H = G_{\{I\}}$ for some invariant $I \subseteq_{\mathrm{end}} M$.


$M \models PA$
$M(0)$
Nonstandard definable elements
$2^{\aleph_0}$ invariant cuts
$\mathbb{N}$

THE AUTOMORPHISM GROUP

Theorem (Kaye)

If $H \triangleleft G$ is closed then $H = G_{\{I\}}$ for some invariant $I \subseteq_{\mathrm{end}} M$.

THE AUTOMORPHISM GROUP

Theorem (Kaye)

If $H \triangleleft G$ is closed then $H = G_{\{I\}}$ for some invariant $I \subseteq_{\text{end}} M$.

Theorem (Kaye, RK, Kotlarski)

$\mathrm{Aut}(\mathbb{Q}, <) \subseteq G \subseteq_{dense} \mathrm{Aut}(\mathbb{Q}, <)$, *but* $G \not\cong \mathrm{Aut}(\mathfrak{M})$ *for any* $\aleph_0$*-categorical countable* $\mathfrak{M}$*, in particular* $G \not\cong \mathrm{Aut}(\mathbb{Q}, <)$.

THE AUTOMORPHISM GROUP

Theorem (Kaye)

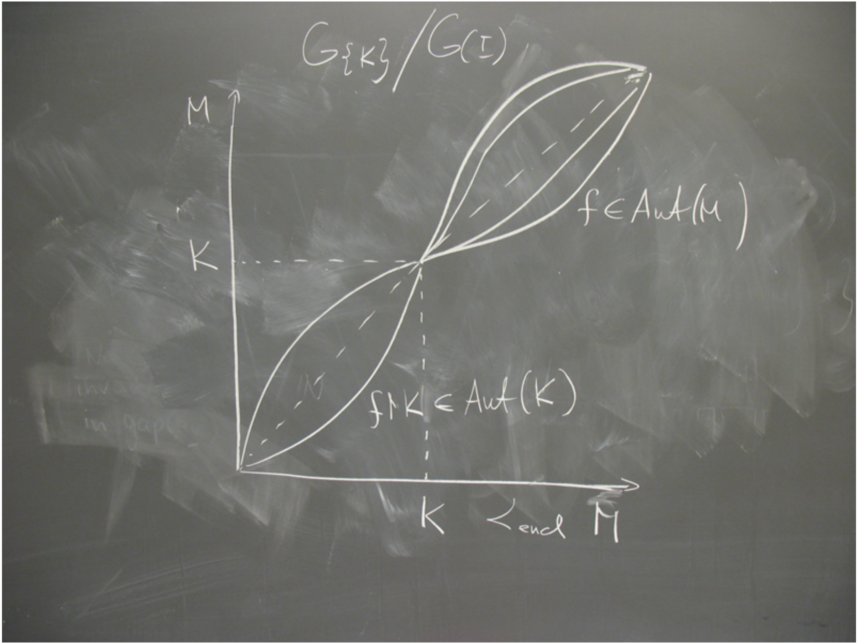
If $H \triangleleft G$ is closed then $H = G_{\{I\}}$ for some invariant $I \subseteq_{\text{end}} M$.

Theorem (Kaye, RK, Kotlarski)

$\mathrm{Aut}(\mathbb{Q},<) \subseteq G \subseteq_{dense} \mathrm{Aut}(\mathbb{Q},<)$, *but* $G \ncong \mathrm{Aut}(\mathfrak{M})$ *for any* $\aleph_0$*-categorical countable* $\mathfrak{M}$*, in particular* $G \ncong \mathrm{Aut}(\mathbb{Q},<)$.

Theorem (Schmerl)

Let $\mathfrak{A}$ be a countable linerly ordered structure. There is $K \prec_{\text{end}} M$ such that $G_{\{K\}}/G_{(K)} \cong \mathrm{Aut}(\mathfrak{A})$.


G{K}/G(I)
M
K
f ∈ Aut(M)
f↾K ∈ Aut(K)
K <end M

THE AUTOMORPHISM GROUP

Theorem (Schmerl)

Let $\mathfrak{A}$ be a countable linerly ordered structure. There is $K \prec_{\mathrm{end}} M$ such that $G_{\{K\}}/G_{(K)} \cong \mathrm{Aut}(\mathfrak{A})$.

THE AUTOMORPHISM GROUP

Theorem (Schmerl)

Let $\mathfrak{A}$ be a countable linerly ordered structure. There is $K \prec_{\text{end}} M$ such that $G_{\{K\}}/G_{(K)} \cong \operatorname{Aut}(\mathfrak{A})$.

Theorem (RK, Kotlarski)

For every $K \prec_{\text{end}} M$

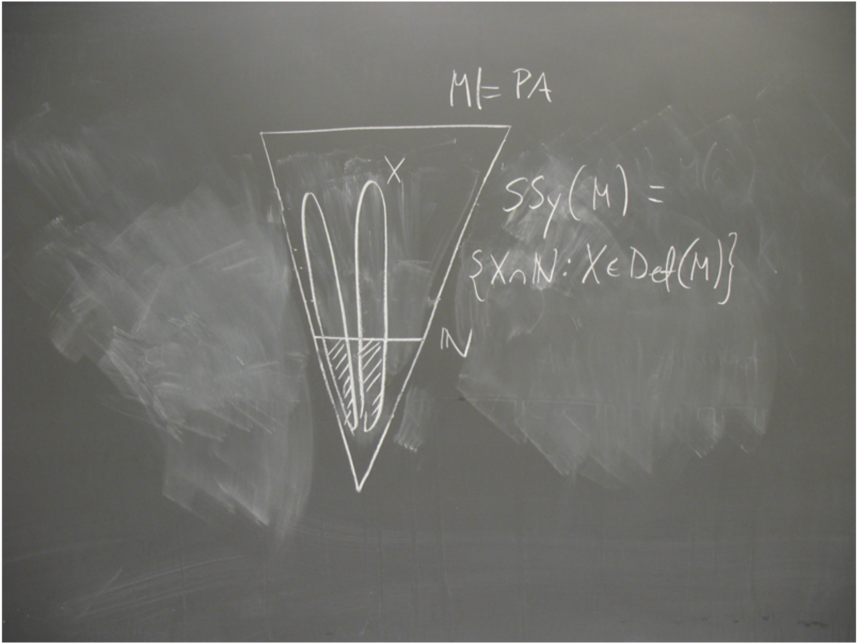
$$|\{g \in \operatorname{Aut}(K) : \forall f \in G_{\{K\}}\ g \neq f \restriction K\}| = 2^{\aleph_0}$$

i.e. many automorphisms of K do not extend to M.

ARITHMETIC SATURATION

Definition

M is arithmetically saturated iff $\mathrm{SSy}(M)$ is arithmetically closed.


M ⊨ PA
X
SSy(M) =
{X ∩ ℕ : X ∈ Def(M)}
ℕ

ARITHMETIC SATURATION

Definition

M is arithmetically saturated iff M is recursively saturated $\mathrm{SSy}(M)$ is arithmetically closed.

ARITHMETIC SATURATION

Definition

M is arithmetically saturated iff M is recursively saturated $\mathrm{SSy}(M)$ *is arithmetically closed.*

Theorem (Scott)

If $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ and $|\mathfrak{X}| = \aleph_0$, then

$$\exists M \models \mathrm{PA},\ \mathfrak{X} = \mathrm{SSy}(M) \Leftrightarrow (\omega, \mathfrak{X}) \models \mathrm{WKL}_0.$$

ARITHMETIC SATURATION

Definition

M is arithmetically saturated iff M is recursively saturated $\mathrm{SSy}(M)$ *is arithmetically closed.*

Theorem (Scott)

If $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ and $|\mathfrak{X}| = \aleph_0$, then

$$\exists M \models \mathrm{PA},\ \mathfrak{X} = \mathrm{SSy}(M) \Leftrightarrow (\omega, \mathfrak{X}) \models \mathrm{WKL}_0.$$

Proposition

If M is recursively saturated, then M is arithmetically saturated iff $(\omega, \mathrm{SSy}(M)) \models \mathrm{ACA}_0$.

ARITHMETIC SATURATION

Theorem (Lascar)

If M is arithmetically saturated, then $H < G$ is open iff $[G : H] < 2^{\aleph_0}$.

ARITHMETIC SATURATION

Theorem (Lascar)

If M is arithmetically saturated, then $H < G$ is open iff $[G : H] < 2^{\aleph_0}$.

Theorem (Kaye, RK, Kotlarski)

M is arithmetically saturated iff M has a maximal automorphism.

ARITHMETIC SATURATION

Theorem (Lascar)

If M is arithmetically saturated, then $H < G$ is open iff $[G : H] < 2^{\aleph_0}$.

Theorem (Kaye, RK, Kotlarski)

M is arithmetically saturated iff M has a maximal automorphism.

Theorem (RK, Schmerl)

M arithmetically saturated iff $\mathrm{cf}(G) > \aleph_0$.

ARITHMETIC SATURATION

Theorem (Lascar)

If M is arithmetically saturated, then $H < G$ is open iff $[G : H] < 2^{\aleph_0}$.

Theorem (Kaye, RK, Kotlarski)

M is arithmetically saturated iff M has a maximal automorphism.

Theorem (RK, Schmerl)

M arithmetically saturated iff $\mathrm{cf}(G) > \aleph_0$.

Theorem (Enayat)

M is arithmetically saturated iff for each $K \prec M$, then there is $f \in G$ such that $\mathrm{fix}(f) \cong K$.

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING SSy(M) IN G.

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING SSy(M) IN G.

Theorem (RK, Schmerl)

If M_0 and M_1 are arithmetically saturated and $M_0 \equiv M_1$, then

$$\mathrm{Aut}(M_0) \cong \mathrm{Aut}(M_1) \text{ iff } \mathrm{SSy}(M_0) = SSy(M_1).$$

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING SSy(M) IN G.

Theorem (RK, Schmerl)

If M_0 and M_1 are arithmetically saturated and $M_0 \equiv M_1$, then

$$\mathrm{Aut}(M_0) \cong \mathrm{Aut}(M_1) \text{ iff } \mathrm{SSy}(M_0) = SSy(M_1).$$

Theorem (Kaye, RK, Kotlarski)

If at least one of M_0 and M_1 is arithmetically saturated, and one is a model of TA*, and the other is not; then*

$$\mathrm{Aut}(M_0) \ncong \mathrm{Aut}(M_1).$$

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING (fragments of) Th(M) IN G.

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING (fragments of) Th(M) IN G.

Theorem (Nurkhaidarov)

There are countable arithmetically saturated M_0, M_1, M_2, M_3 such that for all $i, j < 4$, $\mathrm{SSy}(M_i) = \mathrm{SSy}(M_j)$ and for $i \neq j$

$$\mathrm{Aut}(M_i) \ncong \mathrm{Aut}(M_j).$$

[2] $SSy_0(M) = \mathrm{Rep}(\mathrm{Th}(M))$

THE SPECTRUM OF AUTOMORPHISM GROUPS. CODING (fragments of) Th(M) IN G.

Theorem (Nurkhaidarov)

There are countable arithmetically saturated M_0, M_1, M_2, M_3 such that for all $i, j < 4$, $\mathrm{SSy}(M_i) = \mathrm{SSy}(M_j)$ and for $i \neq j$

$$\mathrm{Aut}(M_i) \ncong \mathrm{Aut}(M_j).$$

Theorem (Nurkhaidarov)

If M_0 and M_1 are arithmetically saturated and $\mathrm{Aut}(M_0) \cong \mathrm{Aut}(M_1)$; then for every $n < \omega$

$$(\omega, \mathrm{SSy}_0(M_0)^2) \models \mathrm{RT}^n_2 \text{ iff } (\omega, \mathrm{SSy}_0(M_1)) \models \mathrm{RT}^n_2.$$

[2] $SSy_0(M) = \mathrm{Rep}(\mathrm{Th}(M))$

THE FOUR MODELS OF NURKHAIDAROV

1. $(\omega, \mathsf{SSy}_0(M_0)) \models \neg\mathsf{RT}^2_2$ (Hirst).

THE FOUR MODELS OF NURKHAIDAROV

1. $(\omega, \mathsf{SSy}_0(M_0)) \models \neg\mathsf{RT}^2_2$ (Hirst).
2. $(\omega, \mathsf{SSy}_0(M_1)) \models \mathsf{RT}^2_2 \wedge \neg\mathsf{RT}^3_2$ (Seetapun, Slaman).

THE FOUR MODELS OF NURKHAIDAROV

1. $(\omega, \mathsf{SSy}_0(M_0)) \models \neg\mathsf{RT}^2_2$ (Hirst).
2. $(\omega, \mathsf{SSy}_0(M_1)) \models \mathsf{RT}^2_2 \wedge \neg\mathsf{RT}^3_2$ (Seetapun, Slaman).
3. $M_2 \not\models \mathsf{TA}$, $(\omega, \mathsf{SSy}_0(M_2))) \models \mathsf{RT}^3_2$ (folklore).

THE FOUR MODELS OF NURKHAIDAROV

1. $(\omega, \mathrm{SSy}_0(M_0)) \models \neg\mathrm{RT}^2_2$ (Hirst).
2. $(\omega, \mathrm{SSy}_0(M_1)) \models \mathrm{RT}^2_2 \wedge \neg\mathrm{RT}^3_2$ (Seetapun, Slaman).
3. $M_2 \not\models \mathrm{TA}$, $(\omega, \mathrm{SSy}_0(M_2))) \models \mathrm{RT}^3_2$ (folklore).
4. $M_3 \models \mathrm{TA}$.

THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS

THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS

Question

Do recursively saturated but not arithmetically saturated models have the small index property?

THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS

Question

Do recursively saturated but not arithmetically saturated models have the small index property?

Question

If M is recursively saturated but not arithmetically saturated, is there $f \in G$ such that $[f]$ is comeager?

THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS

Question

Do recursively saturated but not arithmetically saturated models have the small index property?

Question

If M is recursively saturated but not arithmetically saturated, is there $f \in G$ such that $[f]$ is comeager?

Question

Let T be a completion of PA. *Are there recursively saturated but not arithmetically saturated models M_0 and M_1 of T such that* $\mathrm{Aut}(M_0) \ncong \mathrm{Aut}(M_1)$*?*

Psychologists assure us that tall people command more attention and respect than short ones. In as antropomorphic field as logic, it would follow that taller concepts excite more imagination than shorter ones. Thus, more is written about tall end extensions than stubby cofinal ones, and asked for a preference between tall and short models, most logicians would take the tall choice. Such a high-minded strategy might work well in the short run; but in the long run we must pay everything its due.

C.Smoryński
Cofinal Extensions of Nonstandard Models of Arithmetic, 1981

THE SECOND NEGLECTED CASE: SHORT MODELS

Notation: $M(a) = \sup(\mathrm{Scl}(a)) = \bigcap\{K \prec_{\mathrm{end}} M : a \in K\}$

$$G(a) = \mathrm{Aut}(M(a))$$

$$G \restriction_{M(a)} = \{f \restriction M(a) : f \in G_{\{M(a)\}}\}$$

THE SECOND NEGLECTED CASE: SHORT MODELS

Notation: $M(a) = \sup(\mathrm{Scl}(a)) = \bigcap\{K \prec_{\mathrm{end}} M : a \in K\}$

$$G(a) = \mathrm{Aut}(M(a))$$

$$G \upharpoonright_{M(a)} = \{f \upharpoonright M(a) : f \in G_{\{M(a)\}}\}$$

Theorem (Shochat)

$G \upharpoonright_{M(a)}$ is not normal in $G(a)$, it is dense in $G(a)$, and $[G(a) : G \upharpoonright_{M(a)}] = 2^{\aleph_0}$.

THE SECOND NEGLECTED CASE: SHORT MODELS

Notation: $M(a) = \sup(\mathrm{Scl}(a)) = \bigcap\{K \prec_{\mathrm{end}} M : a \in K\}$

$$G(a) = \mathrm{Aut}(M(a))$$

$$G \restriction_{M(a)} = \{f \restriction M(a) : f \in G_{\{M(a)\}}\}$$

Theorem (Shochat)

$G \restriction_{M(a)}$ is not normal in $G(a)$, it is dense in $G(a)$, and $[G(a) : G \restriction_{M(a)}] = 2^{\aleph_0}$.

Question

Is $G \restriction_{M(a)}$ maximal in $G(a)$?

DIVERSITY AMONG SHORT MODELS

Theorem (Kotlarski)

There are $a_i : i < \omega$ *such that for* $i \neq j$, $M(a_i) \not\cong M(a_j)$.

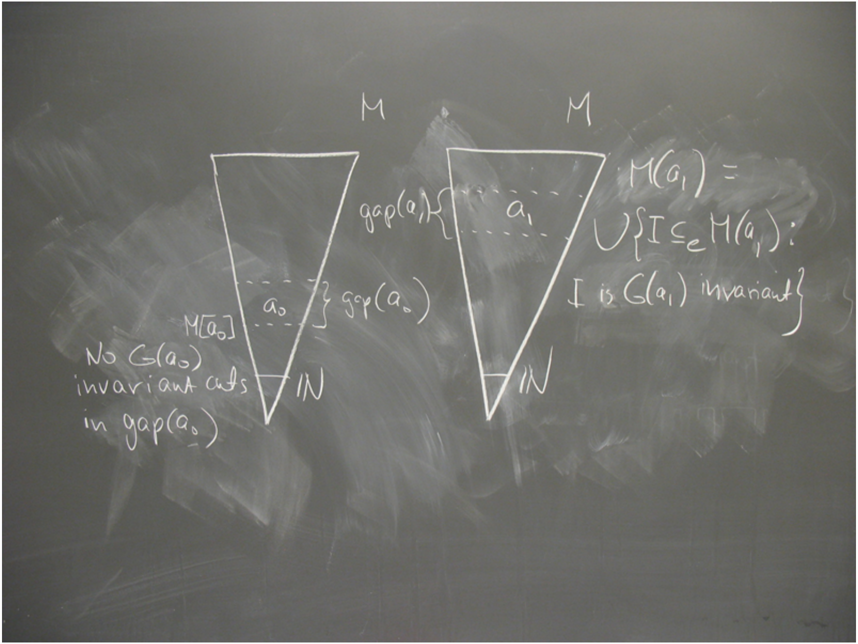
DIVERSITY AMONG SHORT MODELS

Theorem (Kotlarski)

There are $a_i : i < \omega$ such that for $i \neq j$, $M(a_i) \ncong M(a_j)$.

Theorem (Shochat)

There are a_0, a_1 such that $G(a_0) \ncong G(a_1)$ as topological groups.


M
M
$M(a_1) =$
$\bigcup\{I \subseteq_e M(a_1) :$
I is $G(a_1)$ invariant$\}$
$gap(a_1)$
a_1
a_0
$gap(a_0)$
$M[a_0]$
No $G(a_0)$
invariant cuts
in $gap(a_0)$
$\mathbb{N}$
$\mathbb{N}$

TOPOLOGY ON $G(a)$

Theorem (Shochat)

There are a_0, a_1 such that $G(a_0) \not\cong G(a_1)$ as topological groups.

- $\bigcap\{H \triangleleft G(a_0) : H \text{ is closed }\} = G_{\{M[a_0]\}}$
- $\bigcap\{H \triangleleft G(a_1) : H \text{ is closed }\} = \{id\}$

DIGRESSION: STRONG NON-RECONSTRUCTION RESULT

Theorem (Schmerl)

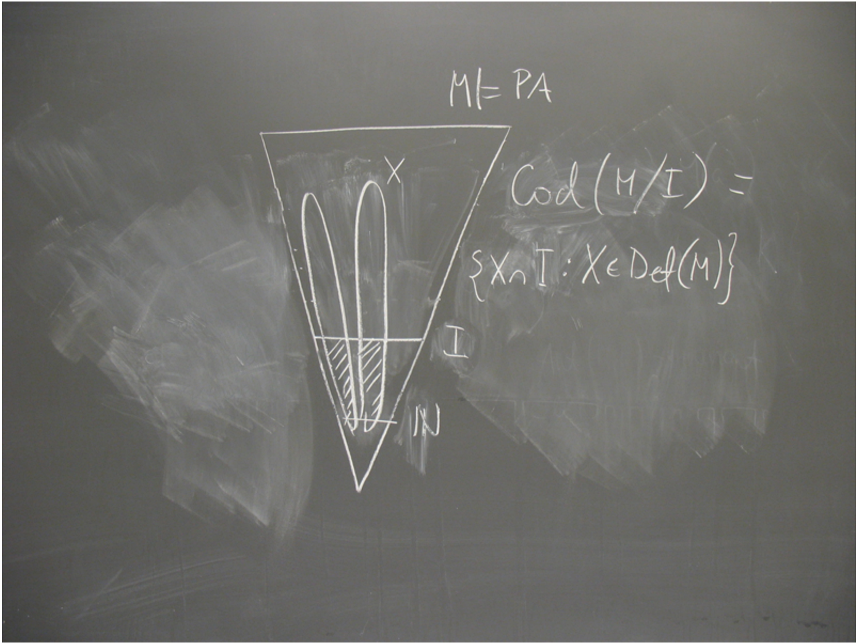
Let $\mathfrak{A}$ be an infinite linearly ordered structure. There is $M \models \mathrm{PA}$, $|\mathfrak{A}| = |M|$ (and M is not recursively saturated) such that $\mathrm{Aut}(\mathfrak{A}) \cong \mathrm{Aut}(M)$.

THE THIRD NEGLECTED CASE: EXTENDING AUTOMORPHISMS TO COFINAL EXTENSIONS

Theorem (RK, Kotlarski)

If $K \prec_{\text{end}} M$, $f \in \mathrm{Aut}(K, \mathrm{Cod}(M/K))$, *then*[3] $f = g \restriction K$ *for some* $g \in \mathrm{Aut}(M)$.

[3] With minor restrictions.


M ⊨ PA
X
Cod(M/I) =
{X ∩ I : X ∈ Def(M)}
I
ℕ

THE THIRD NEGLECTED CASE: EXTENDING AUTOMORPHISMS TO COFINAL EXTENSIONS

Theorem (RK, Kotlarski)

If $K \prec_{\text{end}} M$, $f \in \mathrm{Aut}(K, \mathrm{Cod}(M/K))$, *then*[4] $f = g \restriction K$ *for some* $g \in \mathrm{Aut}(M)$.

[4] With minor restrictions.

THE THIRD NEGLECTED CASE: EXTENDING AUTOMORPHISMS TO COFINAL EXTENSIONS

Theorem (RK, Kotlarski)

If $K \prec_{\text{end}} M$, $f \in \mathrm{Aut}(K, \mathrm{Cod}(M/K))$, *then*[4] $f = g \restriction K$ *for some* $g \in \mathrm{Aut}(M)$.

Theorem (RK, Kotlarski)

If $K \prec_{\text{cof}} M$, $f \in \mathrm{Aut}(K, \mathrm{Cod}(M/K))$, *and the extension* $K \prec_{\text{cof}} M$ *has the the description property, then* $f = g \restriction K$ *for some* $g \in \mathrm{Aut}(M)$.

[4] With minor restrictions.

THE DESCRIPTION PROPERTY

Definition

The extension $K \prec_{\text{cof}} M$ has the description property if for every $a \in M \setminus K$ there is a coded in M nested sequence $\langle A_i : i < \omega \rangle$ of K-finite sets such that

1. $M \models a \in A_i$ for all $i < \omega$;
2. For each K-finite B such that $a \in B$, there is an $i < \omega$ such that $A_i \subseteq B$.

Theorem (RK, Kotlarski)

For each M, there are K_0, K_1 such that $K_0 \prec_{\text{cof}} M \prec_{\text{cof}} K_1$ and both extensions have the description property.

Question

For a given M is there an N such that $M \prec_{\mathrm{cof}} N$ has the description property and $\mathrm{SSy}(M) = \mathrm{SSy}(N)$?

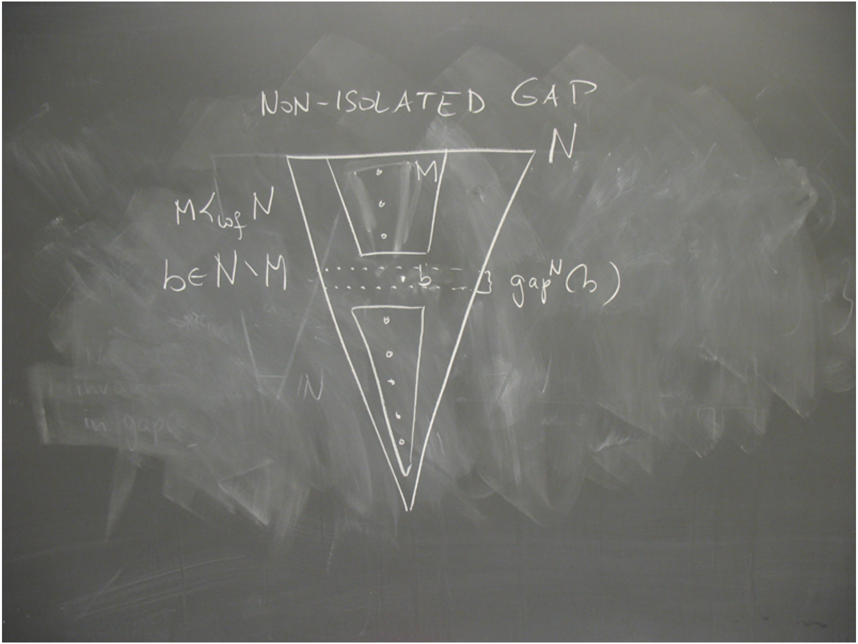
Question

For a given M, is there an N such that for each bounded $A \in \mathrm{Cod}(N/M)$ there is a $b \in M$ such that b codes A?

ISOLATED GAPS

Definition

If $M \prec_{\mathrm{cof}} N$ and $b \in N \setminus M$, then $\mathrm{gap}^N(b)$ is non-isolated if there are $d < \mathrm{gap}(b) < e \in N$ such that $[d, e] \cap M = \emptyset$


NON-ISOLATED GAP
N
M
M <_wf N
b ∈ N \ M
b
gap^N(b)

ISOLATED GAPS

Theorem (RK, Kotlarski)

Every cofinal extension has non-isolated gaps.

ISOLATED GAPS

Theorem (RK, Kotlarski)

Every cofinal extension has non-isolated gaps.

Theorem (RK, Kotlarski)

No extension with the description property has isolated gaps.

ISOLATED GAPS

Theorem (RK, Kotlarski)

Every cofinal extension has non-isolated gaps.

Theorem (RK, Kotlarski)

No extension with the description property has isolated gaps.

Question

Are there cofinal extensions with isolated gaps?