Automorphisms of models of PA: The neglected cases.

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In memory of Henryk Kotlarski 1949-2008
OVERVIEW

I Models of PA in the 1980s
II Automorphisms of models of PA
III Automorphism groups of models of PA
IV The neglected cases. What is left to be done?
MODEL THEORY OF PA IN 1980s

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- Kotlarski, Krajewski, Lachlan, Murawski, Smith: Nonstandard satisfaction classes.
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MODEL THEORY OF PA IN 1980s

- Kotlarski, Krajewski, Lachlan, Murawski, Smith: Nonstandard satisfaction classes.
- Harnik, Knight, Marker, Nadel, Pabion, Richard, Solovay: Turing degrees. Scott sets. Presburger Arithmetic. Reducts. Other...
Let me begin with the pre-history of the subject. The question whether PA has a model with a nontrivial automorphism (i.e., such that its elements cannot be individualized) was due to Hasenjäger. It was solved positively by Ehrenfeucht and Mostowski [3]. Their result and the idea of indiscernibility is nowadays so well known that Hodges [5] writes “today model theorists use it at least once a week,” so let me omit the statement of the Ehrenfeucht-Mostowski Theorem.
Let me begin with the pre-history of the subject. The question whether PA has a model with a nontrivial automorphism (i.e., such that its elements cannot be individualized) was due to Hasenjäger. It was solved positively by Ehrenfeucht and Mostowski [3]. Their result and the idea of indiscernibility is nowadays so well known that Hodges [5] writes “today model theorists use it at least once a week,” so let me omit the statement of the Ehrenfeucht-Mostowski Theorem.
THREE PAPERS ON EXTENDING AUTOMORPHISMS


AUTOMORPHISMS OF COUNTABLE RECURSIVELY SATURATED STRUCTURES

Let $\mathcal{M}$ be countable and recursively saturated.

**Proposition**

For every infinite $A \in \text{Def}(\mathcal{M})$ there are $a, b \in A$ such that $a \neq b$ and $\text{tp}(a) = \text{tp}(b)$.

**Proposition**

For all $a, b \in \mathcal{M}$, if $\text{tp}(a) = \text{tp}(b)$, then there is $f \in \text{Aut}(\mathcal{M})$ such that $f(a) = b$.

**Proposition**

$|\text{Aut}(\mathcal{M})| = 2^{\aleph_0}$. 
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**Proposition**

|$|\text{Aut}(\mathcal{M})| = 2^{\aleph_0}$. 
WHAT IS SPECIAL ABOUT PA?
EHREFEUCHT-GAIFMAN LEMMA

Theorem
If $M \models PA$, $f \in \text{Aut}(M)$, and $f(a) \neq a$, then $f(a) \notin Scl(a)^1$.

1$Scl(a)$ is the definable closure of $a$ in $M$. 
AUTOMORPHIC IMAGES

Definition
For $X \subseteq \mathcal{M}$, $\alpha(X) = |\{f(X) : f \in \text{Aut}(\mathcal{M})\}|$

Basic facts:
AUTOMORPHIC IMAGES

Definition

For $X \subseteq M$, $a(X) = |\{ f(X) : f \in \text{Aut}(M) \}|$

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- $X \in \text{Def}_0(M) \Rightarrow a(X) = 1$
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AUTOMORPHIC IMAGES

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\[
\begin{align*}
X \in \text{Def}_0(\mathcal{M}) & \Rightarrow a(X) = 1 \\
X \in \text{Def}(\mathcal{M}) & \Rightarrow a(X) \leq \aleph_0 \\
a(X) > \aleph_0 & \Rightarrow a(X) = 2^{\aleph_0} \quad \text{(Kueker-Reyes)}
\end{align*}
\]
$a(X)$ IN MODELS OF PA

Assumption from now on: $M \models \text{PA}$ is countable and recursively saturated
\[ a(X) \text{ IN MODELS OF PA} \]

Assumption from now on: \( M \models \text{PA is countable and recursively saturated} \)

Theorem (Schmerl)

a. \( \forall X \subseteq M, a(X) \in \{1, \aleph_0, 2^{\aleph_0}\} \)
a(X) IN MODELS OF PA

Assumption from now on: \( M \models \text{PA} \) is countable and recursively saturated

Theorem (Schmerl)

a. \( \forall X \subseteq M, \ a(X) \in \{1, \aleph_0, 2^{\aleph_0}\} \)

b. \( \forall X \in \text{Def}(M) \setminus \text{Def}_0(M), \ a(X) = \aleph_0. \)
MANY AUTOMORPHIC IMAGES
MANY AUTOMORPHIC IMAGES

Theorem (RK, Kotlarski)

\((M, X) \models \text{PA}^* \text{ and } X \not\in \text{Def}(M) \Rightarrow a(X) = 2^{\aleph_0}\)
MANY AUTOMORPHIC IMAGES

Theorem (RK, Kotlarski)

$(M, X) \models \text{PA}^*$ and $X \notin \text{Def}(M) \implies \alpha(X) = 2^{\aleph_0}$

Theorem

Let $I \subseteq_{\text{end}} M$. Then $\alpha(I) < 2^{\aleph_0}$ iff $\exists a \in M$

\[ I = \sup[Scl(a) \cap I] \text{ or } I = \inf[Scl(a) \cap (M \setminus I)] \]
\[ M = PA \]

\[ Sd(a) < M \]
\[ M| = \text{PA} \]

\[ \exists^H \sup (I) \]

\[ I \preceq \text{end} \ Sd(a) \]

\[ Sd(a) < M \]
MANY AUTOMORPHIC IMAGES

From now on $M \models \text{PA}$ is countable and recursively saturated

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Theorem (RK, Kotlarski, independently Tzouvaras)
$f \in \text{Aut}(M) \text{ and } f \neq id \Rightarrow |\{f^g : g \in \text{Aut}(M)\}| = \alpha(f) = 2^\aleph_0$
MOVING GAPS LEMMA

Theorem (Kotlarski)

If $f \in \text{Aut}(M) \setminus \{id\}$, then for arbitrarily large $a \in M$,

$$f(a) \not\in \text{gap}(a).$$
$M I = P A$

$\sup(Sd(a)) = M(a)$
\[ M = P A \]

\[ \text{sup}(Scl(a)) = M(a) = \bigcap \{ K < H : a \in K \} \]
ML = PA

\[ \bigcup \{ K \mid \forall k \in K : a \notin K \} \cap \bigcup \{ K \mid a \in K \} \]
\[ M[a] \]

\[ \text{gap}(a) = M(a) \setminus M[a] \]
$f \in \text{Aut}(H)$

$M = PA$
$Ml = PA$

$\forall \alpha \in \text{Aut}(H)$

$s \rightarrow \alpha' \quad \{ \text{gap}(\alpha) \}$

$s' \rightarrow s''$
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Corollary
If \( p(v) \) is an unbounded type realized in \( M \), \( f \in \text{Aut}(M) \setminus \{id\} \),
then there is \( a \in p^M \) such that \( f(a) \neq a \).
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Corollary
If \( K \preceq M \) and \( K \cap \text{gap}(a) \neq \emptyset \) for all \( a \in M \), then \( K = M \).
THE AUTOMORPHISM GROUP

$M \models \text{PA countable and recursively saturated}, \ G = \text{Aut}(M)$.

**Theorem (Kaye)**

*If $H \triangleleft G$ is closed then $H = G_{\{I\}}$ for some invariant $I \subseteq_{\text{end}} M$.*
Nonstandard definable elements

$M[0] \prec PA$

$2^{-\delta}$ invariant cuts
THE AUTOMORPHISM GROUP

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Theorem (Kaye, RK, Kotlarski)
$\text{Aut}(\mathbb{Q}, <) \subseteq G \subseteq_{\text{dense}} \text{Aut}(\mathbb{Q}, <)$, but $G \not\cong \text{Aut}(\mathcal{M})$ for any $\aleph_0$-categorical countable $\mathcal{M}$, in particular $G \not\cong \text{Aut}(\mathbb{Q}, <)$. 
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Theorem (Schmerl)

Let $\mathcal{A}$ be a countable linearly ordered structure. There is $K \triangleleft_{\text{end}} M$ such that $G_{\{K\}}/G_{(K)} \cong \text{Aut}(\mathcal{A})$. 
\[ \frac{G_{K^3}}{G(I)} \]

Graph with axes labeled \( M \) and \( K \), with arrows indicating direction. The text on the graph includes:

- \( f \in \text{Aut}(M) \)
- \( f_{|K} \in \text{Aut}(K) \)

Additional notes:

- Insert in gap
- Invariant
THE AUTOMORPHISM GROUP

Theorem (Schmerl)

Let \( \mathcal{A} \) be a countable linearly ordered structure. There is \( K \preceq_{\text{end}} M \) such that \( G_{\{K\}}/G(K) \cong \text{Aut}(\mathcal{A}) \).
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Let $\mathbb{A}$ be a countable linearly ordered structure. There is $K \preceq_{\text{end}} M$ such that $G_{\{K\}} / G(K) \cong \text{Aut}(\mathbb{A})$.

Theorem (RK, Kotlarski)
For every $K \preceq_{\text{end}} M$

$$|\{g \in \text{Aut}(K) : \forall f \in G_{\{K\}} \ g \neq f \restriction K\}| = 2^{\aleph_0}$$

i.e. many automorphisms of $K$ do not extend to $M$. 
ARITHMETIC SATURATION

Definition

$M$ is arithmetically saturated iff $SSy(M)$ is arithmetically closed.
\[ M \subseteq PA \]

\[ SSy(M) = \{ X \cap N : X \in \text{Def}(M) \} \]
ARITHMETIC SATURATION

Definition

$M$ is \textit{arithmetically saturated} iff $M$ is recursively saturated $\text{SSy}(M)$ is arithmetically closed.

Theorem (Scott)

If $X \subseteq \mathcal{P}(\mathbb{N})$ and $|X| = \aleph_0$, then

$\exists M | M = \mathbb{PA}, X = \text{SSy}(M) \iff (\omega, X) | = \text{WKL}_0$.

Proposition

If $M$ is recursively saturated, then $M$ is arithmetically saturated iff $(\omega, \text{SSy}(M)) | = \mathbb{ACA}_0$.
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*M is arithmetically saturated* iff *M is recursively saturated* \( \text{SSy}(M) \) is arithmetically closed.

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ARITHMETIC SATURATION

Theorem (Lascar)

*If* $M$ *is arithmetically saturated, then* $H < G$ *is open iff* $[G : H] < 2^{\aleph_0}$.  

Theorem (Kaye, RK, Kotlarski)

$M$ *is arithmetically saturated iff* $M$ *has a maximal automorphism.*

Theorem (RK, Schmerl)

$M$ *arithmetically saturated iff* $\text{cf}(G) > \aleph_0$.

Theorem (Enayat)

$M$ *is arithmetically saturated iff for each* $K \prec M$, *then there is* $f \in G$ *such that* $\text{fix}(f) \sim = K$.  

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$M$ is arithmetically saturated iff for each $K \prec M$, then there is $f \in G$ such that $\text{fix}(f) \cong K$. 
THE SPECTRUM OF AUTOMORPHISM GROUPS.
CODING SSy(M) IN G.
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Theorem (RK, Schmerl)
If $M_0$ and $M_1$ are arithmetically saturated and $M_0 \equiv M_1$, then
\[ \text{Aut}(M_0) \cong \text{Aut}(M_1) \text{ iff } \text{SSy}(M_0) = \text{SSy}(M_1). \]
Theorem (RK, Schmerl)

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Theorem (Kaye, RK, Kotlarski)

If at least one of $M_0$ and $M_1$ is arithmetically saturated, and one is a model of TA, and the other is not; then

$$\text{Aut}(M_0) \not\cong \text{Aut}(M_1).$$
THE SPECTRUM OF AUTOMORPHISM GROUPS.
CODING (fragments of) Th(M) IN G.
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Theorem (Nurkhaidarov)
There are countable arithmetically saturated \( M_0, M_1, M_2, M_3 \) such that for all \( i, j < 4 \), \( \text{SSy}(M_i) = \text{SSy}(M_j) \) and for \( i \neq j \)

\[
\text{Aut}(M_i) \nRightarrow \text{Aut}(M_j).
\]

\(^2\text{SSy}_0(M) = \text{Rep}(\text{Th}(M))\)
Theorem (Nurkhaidarov)

There are countable arithmetically saturated $M_0, M_1, M_2, M_3$ such that for all $i, j < 4$, $SSy(M_i) = SSy(M_j)$ and for $i \neq j$

\[ Aut(M_i) \not\cong Aut(M_j). \]

Theorem (Nurkhaidarov)

If $M_0$ and $M_1$ are arithmetically saturated and $Aut(M_0) \cong Aut(M_1)$; then for every $n < \omega$

\[ (\omega, SSy_0(M_0)^2) \models RT^2_n \text{ iff } (\omega, SSy_0(M_1)) \models RT^2_n. \]

$^2SSy_0(M) = \text{Rep}(\text{Th}(M))$
THE FOUR MODELS OF NURKHAIDAROV

1. $(\omega, Ssy_0(M_0)) \models \neg RT_2$ (Hirst).
THE FOUR MODELS OF NURKHAIDAROV

1. \((\omega, \text{SSy}_0(M_0)) \models \neg\text{RT}_2^2\) (Hirst).

2. \((\omega, \text{SSy}_0(M_1)) \models \text{RT}_2^2 \land \neg\text{RT}_2^3\) (Seetapun, Slaman).
THE FOUR MODELS OF NURKHAIDAROV

1. \((\omega, \text{SSy}_0(M_0)) \models \neg \text{RT}^2_2\) (Hirst).
2. \((\omega, \text{SSy}_0(M_1)) \models \text{RT}^2_2 \land \neg \text{RT}^3_2\) (Seetapun, Slaman).
3. \(M_2 \not\models \text{TA}\), \((\omega, \text{SSy}_0(M_2))) \models \text{RT}^3_2\) (folklore).
THE FOUR MODELS OF NURKHAI DAROV

1. \((\omega, SSy_0(M_0)) \models \neg RT^2_2\) (Hirst).
2. \((\omega, SSy_0(M_1)) \models RT^2_2 \land \neg RT^3_2\) (Seetapun, Slaman).
3. \(M_2 \not\models TA, \ (\omega, SSy_0(M_2))) \models RT^3_2\) (folklore).
4. \(M_3 \models TA\).
THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS
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Question

Do recursively saturated but not arithmetically saturated models have the small index property?
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Do recursively saturated but not arithmetically saturated models have the small index property?

Question
If $M$ is recursively saturated but not arithmetically saturated, is there $f \in G$ such that $\lbrack f \rbrack$ is comeager?
THE FIRST NEGLECTED CASE: NOT ARITHMETICALLY SATURATED MODELS

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Do recursively saturated but not arithmetically saturated models have the small index property?

Question

If $M$ is recursively saturated but not arithmetically saturated, is there $f \in G$ such that $[f]$ is comeager?

Question

Let $T$ be a completion of PA. Are there recursively saturated but not arithmetically saturated models $M_0$ and $M_1$ of $T$ such that $\text{Aut}(M_0) \not\cong \text{Aut}(M_1)$?
Psychologists assure us that tall people command more attention and respect than short ones. In as antropomorphic field as logic, it would follow that taller concepts excite more imagination than shorter ones. Thus, more is written about tall end extensions than stubby cofinal ones, and asked for a preference between tall and short models, most logicians would take the tall choice. Such a high-minded strategy might work well in the short run; but in the long run we must pay everything its due.

C.Smoryński

Cofinal Extensions of Nonstandard Models of Arithmetic, 1981
THE SECOND NEGLECTED CASE: SHORT MODELS

Notation: \( M(a) = \sup(\text{Scl}(a)) = \bigcap\{K \prec_{\text{end}} M : a \in K\} \)

\[ G(a) = \text{Aut}(M(a)) \]

\[ G \upharpoonright M(a) = \{f \upharpoonright M(a) : f \in G_{\{M(a)\}}\} \]
THE SECOND NEGLECTED CASE: SHORT MODELS

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\[ G(a) = \text{Aut}(M(a)) \]

\[ G \upharpoonright_{M(a)} = \{f \upharpoonright_{M(a)} : f \in G\{M(a)\}\} \]

Theorem (Shochat)

\( G \upharpoonright_{M(a)} \) is not normal in \( G(a) \), it is dense in \( G(a) \), and

\[ [G(a) : G \upharpoonright_{M(a)}] = 2^{\aleph_0}. \]
THE SECOND NEGLECTED CASE: SHORT MODELS

Notation: $M(a) = \sup(\text{Scl}(a)) = \bigcap\{K \prec_{\text{end}} M : a \in K\}$

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Theorem (Shochat)

$G \upharpoonright M(a)$ is not normal in $G(a)$, it is dense in $G(a)$, and $[G(a) : G \upharpoonright M(a)] = 2^{\aleph_0}$.

Question

Is $G \upharpoonright M(a)$ maximal in $G(a)$?
DIVERSITY AMONG SHORT MODELS

Theorem (Kotlarski)
There are $a_i : i < \omega$ such that for $i \neq j$, $M(a_i) \not\cong M(a_j)$. 
DIVERSITY AMONG SHORT MODELS

Theorem (Kotlarski)
There are $a_i : i < \omega$ such that for $i \neq j$, $M(a_i) \not\cong M(a_j)$.

Theorem (Shochat)
There are $a_0, a_1$ such that $G(a_0) \not\cong G(a_1)$ as topological groups.
\[ M(a_0) \]
\[ \text{No } G(a_0) \text{ invariant cuts in } \text{gap}(a_0) \]
\[ M(a_1) = \bigcup \{ I \subseteq M(a_1) : I \text{ is } G(a_1) \text{ invariant} \} \]
Theorem (Shochat)

There are $a_0, a_1$ such that $G(a_0) \not\cong G(a_1)$ as topological groups.

$$\bigcap \{ H \triangleleft G(a_0) : H \text{ is closed} \} = G\{M[a_0]\}$$

$$\bigcap \{ H \triangleleft G(a_1) : H \text{ is closed} \} = \{id\}$$
Theorem (Schmerl)

Let $\mathcal{A}$ be an infinite linearly ordered structure. There is $M \models \text{PA}$, $|\mathcal{A}| = |M|$ (and $M$ is not recursively saturated) such that $\text{Aut}(\mathcal{A}) \cong \text{Aut}(M)$. 
THE THIRD NEGLECTED CASE: EXTENDING AUTOMORPHISMS TO COFINAL EXTENSIONS

Theorem (RK, Kotlarski)
If $K \prec_{\text{end}} M$, $f \in \text{Aut}(K, \text{Cod}(M/K))$, then$^3$ $f = g \upharpoonright K$ for some $g \in \text{Aut}(M)$.

$^3$With minor restrictions.
\[ M/I = \text{PA} \]

\[ \text{Cod}(M/I) = \{ x \in T : x \in \text{Def}(M) \} \]
Theorem (RK, Kotlarski)

If $K \prec_{\text{end}} M$, $f \in \text{Aut}(K, \text{Cod}(M/K))$, then\(^4\) $f = g \upharpoonright K$ for some $g \in \text{Aut}(M)$.

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If $K \prec_{\text{end}} M$, $f \in \text{Aut}(K, \text{Cod}(M/K))$, then $^4 f = g \upharpoonright K$ for some $g \in \text{Aut}(M)$.

Theorem (RK, Kotlarski)
If $K \prec_{\text{cof}} M$, $f \in \text{Aut}(K, \text{Cod}(M/K))$, and the extension $K \prec_{\text{cof}} M$ has the the description property, then $f = g \upharpoonright K$ for some $g \in \text{Aut}(M)$.

$^4$With minor restrictions.
THE DESCRIPTION PROPERTY

Definition
The extension $K \prec_{\text{cof}} M$ has the description property if for every $a \in M \setminus K$ there is a coded in $M$ nested sequence $\langle A_i : i < \omega \rangle$ of $K$-finite sets such that

1. $M \models a \in A_i$ for all $i < \omega$;
2. For each $K$-finite $B$ such that $a \in B$, there is an $i < \omega$ such that $A_i \subseteq B$.

Theorem (RK, Kotlarski)
For each $M$, there are $K_0, K_1$ such that $K_0 \prec_{\text{cof}} M \prec_{\text{cof}} K_1$ and both extensions have the description property.
Question

For a given $M$ is there an $N$ such that $M \prec_{\text{cof}} N$ has the description property and $\text{SSy}(M) = \text{SSy}(N)$?

Question

For a given $M$, is there an $N$ such that for each bounded $A \in \text{Cod}(N/M)$ there is a $b \in M$ such that $b$ codes $A$?
Definition

If $M \prec \text{cof} \ N$ and $b \in N \setminus M$, then $\text{gap}^N(b)$ is non-isolated if there are $d < \text{gap}(b) < e \in N$ such that $[d, e] \cap M = \emptyset$
NON-ISOLATED GAP

M \not\sqsubseteq N

b \in N \setminus M

\text{in gaps in } N

\text{in gaps in gap} N(b)
ISOLATED GAPS

Theorem (RK, Kotlarski)

*Every cofinal extension has non-isolated gaps.*
ISOLATED GAPS

Theorem (RK, Kotlarski)
Every cofinal extension has non-isolated gaps.

Theorem (RK, Kotlarski)
No extension with the description property has isolated gaps.
ISOLATED GAPS

Theorem (RK, Kotlarski)
Every cofinal extension has non-isolated gaps.

Theorem (RK, Kotlarski)
No extension with the description property has isolated gaps.

Question
Are there cofinal extensions with isolated gaps?