SOME MODEL THEORY OF POLISH STRUCTURES

1. General goal

Apply ideas and techniques from stable theories to purely topological objects, e.g. to Polish G-spaces.

Main idea:

Define a topological notion of independence that has similar properties to those of forking independence in stable theories.

Introduce notions of 'definable' sets, imaginary sorts, etc., so that you can 'think' about them as in model theory (although they are completely different notions, defined in a purely topological way).

2. Profinite structures (Newelski)

Easy example:

$$\mathbb{Z}_2^\omega = \varprojlim \mathbb{Z}_2^n$$

$(\mathbb{Z}_2^\omega, Aut^0(\mathbb{Z}_2^\omega))$ is a profinite group regarded as profinite structure.

Orbits on $\mathbb{Z}_2^\omega$ under $Aut^0(\mathbb{Z}_2^\omega)$:

$(\mathbb{Z}_2^\omega, Aut^0(\mathbb{Z}_2^\omega))$ is small and m-stable.

Original definition:

$X = \varprojlim X_i$ – a profinite topological space

$Homeo^0(X)$ - the group of all homeomorphisms of X respecting the inverse system defining X:

$f \in Homeo^0(X) \iff f$ is induced by an automorphism of the inverse system $\langle X_i, f_{ji} \rangle_{i \leq j}$

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "f_{10}"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "f_{21}"'] \arrow[d, "f_2"'] & \cdots \arrow[l, "f_{32}"'] & X \arrow[d, "f"'] \\ X_0 & X_1 \arrow[l, "f_{10}"'] & X_2 \arrow[l, "f_{21}"'] & \cdots \arrow[l, "f_{32}"'] & X \end{tikzcd}$$

$Homeo^0(X) \subseteq_c X^X$

$Aut^*(X)$ - a closed subgroup of $Homeo^0(X)$

$(X, Aut^*(X))$ - a profinite structure

$Aut^*(X)$ - the structural group of X

$Homeo^0(X)$ - the standard structural group of X

3. Compact structures

Definition A compact structure is a pair (X, G) where G is a compact group acting continuously and faithfully on a compact metric space X.

Proposition (K) A compact structure (X, G) is a profinite structure iff X is a profinite space.

4. Main definitions

$(X, Aut^*(X))$ - a profinite structure
$a \in X^n$
A - a finite subset of X

$o(a/A) := \{f(a) : f \in Aut^*(X/A)\}$ - the orbit of a over A

$(X, Aut^*(X))$ is small $\iff$ there are countably many orbits over any finite set $\iff$ for every $n \in \omega$ there are countably many orbits on X^n under $Aut^*(X)$.

$(X, Aut^*(X))$ is m-stable $\iff$ there are no $a \in X$ and finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq X$ such that $o(a/A_{i+1}) \subseteq_{nwd} o(a/A_i)$ for every $i \in \omega$.

5. Small profinite groups

Remark A profinite group has a basis of open neighborhoods of identity consisting of clopen invariant normal subgroups.

Remark Each small abelian profinite group has a finite exponent.

$\{X_i, f_{ji} : i, j \in \omega, i \leq j\}$ - an inverse system of finite abelian groups

$$X = \varprojlim X_i$$

Theorem (K) If X has a finite exponent, then $(X, Aut^0(X))$ is small and m-stable.

The proof uses model theory of modules.

Main Conjecture (Newelski) Each small profinite group has an open abelian subgroup.

Theorem (Wagner) Each small, m-stable profinite group has an open abelian subgroup.

6. m-independence

$(X, Aut^*(X))$ – a profinite structure
a – a finite tuple of elements of X
A, B – finite subsets of X

$a \overset{m}{\perp}_A B \iff o(a/AB)$ is open in $o(a/A)$

$a \overset{m}{\not\perp}_A B \iff o(a/AB)$ is nowhere dense in $o(a/A)$

$a \in acl(A) \iff o(a/A)$ is finite

Theorem (Newelski) Let $(X, Aut^*(X))$ be a small profinite structure. Then:

(1) (Symmetry) For every finite $A, B, C \subseteq X$ we have that $A \overset{m}{\perp}_C B$ iff $B \overset{m}{\perp}_C A$.

(2) (Transitivity) For every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$ we have that $a \overset{m}{\perp}_A C$ iff $a \overset{m}{\perp}_B C$ and $a \overset{m}{\perp}_A B$.

(3) For every finite $A \subseteq X$, $a \in acl(A)$ iff $a \overset{m}{\perp}_A B$ for all finite $B \subseteq X$.

(4) (Extensions) For every finite $a, A, B \subseteq X$ there is some $b \in o(a/A)$ with $b \overset{m}{\perp}_A B$.

Newelski also introduced the notion of definable (i.e. closed and invariant over a finite subset) set and imaginary sort (i.e. quotient by a definable equivalence relation).

Using (1), (2), (3), (4) and smallness, he developed a counterpart of geometric stability theory for small profinite structures. The deepest result seems to be the group configuration theorem.

Investigations of small profinite structures:

1. Examples (K, N).
2. Model theory (K, **N**, W).
3. Structure of groups and rings (K, N, W).
4. Interpretability in first order structures, e.g in fields (K).
5. Applications to the pure theory of profinite groups (???).

Problem There are not many explicit examples of small profinite structures.

7. Polish structures: definition

Definition A Polish structure is a pair (X, G) where G is a Polish group acting (faithfully) on a set X so that the stabilizers of all elements of X are closed.

Examples: profinite structures, Polish G-spaces, Borel G-spaces

Examples of small Polish structures:

1. $(S^n, Homeo(S^n))$
2. $(I^\omega, Homeo(I^\omega))$
3. $(P, Homeo(P))$ where P is the pseudo-arc

Problem $\overset{m}{\smile\!\!\!\!\!\mid}$ does not make sense as there is no topology on X. Even if (X, G) is a Polish G-space, orbits are not necessarily closed; they can be even meager in their relative topologies. So $\overset{m}{\smile\!\!\!\!\!\mid}$ is not a good relation of independence in Polish structures.

8. Non-meager independence $\overset{nm}{\downfree}$

(X, G) – a Polish structure
a – a finite tuple of elements of X
A, B – finite subsets of X;
$\pi_{A,a} : G_A \to o(a/A)$ is given by $\pi_{A,a}(g) = ga$

$$a \overset{nm}{\downfree}_A B \iff \pi_{A,a}^{-1}[o(a/AB)] \subseteq_{nm} \pi_{A,a}^{-1}[o(a/A)]$$
$$a \overset{nm}{\not\downfree}_A B \iff \pi_{A,a}^{-1}[o(a/AB)] \subseteq_{m} \pi_{A,a}^{-1}[o(a/A)]$$

Remark $a \overset{nm}{\downfree}_A B$ iff $G_{AB}G_{Aa} \subseteq_{nm} G_A$.

Theorem (K) $\overset{nm}{\downfree} \models (1), (2), (3)$. If (X, G) is small, then $\overset{nm}{\downfree} \models (4)$.

(1) is trivial; (3) and (4) are easy.

The proof of (2) is more complicated. It uses some descriptive set theory (e.g. a modification of the proof of Effros theorem, Kuratowski-Ulam theorem, the fact that analytic sets have Baire property). The following lemma is the main ingredient.

Lemma (K) Suppose that H_1 and H_2 are closed subgroups of a Polish group H such that H_1H_2 is non-meager in its relative topology. Let $H_3 = H_1 \cap H_2$. Then for every analytic sets $A_1 = A_1H_3 \subseteq_{nm} H_1$ and $A_2 = H_3A_2 \subseteq_{nm} H_2$ we have $A_1A_2 \subseteq_{nm} H_1H_2$.

Theorem (K) Let (X, G) be a Polish structure such that G acts continuously on a Hausdorff space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology. Then $a \overset{nm}{\perp}_A B \iff o(a/AB) \subseteq_{nm} o(a/A)$.

Corollary In every compact structure, $\overset{m}{\perp} = \overset{nm}{\perp}$.

One can also define collections of 'definable' subsets and imaginary sorts.

$D \subseteq X^n$ is definable $\iff$ D is invariant over some finite $A \subseteq X$ and $Stab(D) <_c G$.

Imaginary sorts: the sets X^n/E where E is any invariant equivalence relation on X^n whose all classes have closed stabilizers.

9. $\mathcal{NM}$-rank

(X, G) – a Polish structure

O – the set of all orbits over finite sets

We can measure a topological complication of orbits by means of the function

$\mathcal{NM} : O \to Ord \cup \{\infty\}$ defined by

$\mathcal{NM}(a/A) \geq \alpha + 1 \iff$ there is a finite set $B \supseteq A$ such that $a \overset{nm}{\not\perp}_A B$ and $\mathcal{NM}(a/B) \geq \alpha$.

Lascar inequalities:

$\mathcal{NM}(a/bA) + \mathcal{NM}(b/A) \leq \mathcal{NM}(ab/A)$
$\mathcal{NM}(ab/A) \leq \mathcal{NM}(a/bA) \oplus \mathcal{NM}(b/A)$.

$\mathcal{NM}$-gap conjecture: $\mathcal{NM}(o) \in \omega \cup \{\infty\}$ for every orbit $o \in O$.

Definition (X, G) is nm-stable if every 1-orbit (n-orbit) has ordinal $\mathcal{NM}$-rank.

Equivalently, there are no finite sets $A_0 \subseteq A_1 \subseteq \ldots \subseteq \omega$ and $a \in X$ such that $a \overset{nm}{\not\perp}_{A_i} A_{i+1}$.

$\mathcal{NM}(X) := sup\{\mathcal{NM}(a) : a \in X\}$.

Examples:

1. In $(\mathbb{Z}_2^\omega, Aut^0(\mathbb{Z}_2^\omega))$ we get $\mathcal{NM}(\mathbb{Z}_2^\omega) = 1$.
2. In $(\mathbb{Z}_{p^n}^\omega, Aut^0(\mathbb{Z}_{p^n}^\omega))$ we get $\mathcal{NM}(\mathbb{Z}_{p^n}^\omega) = n$.
3. In $(S^n, Homeo(S^n))$ we get $\mathcal{NM}(S^n) = 1$.
4. In $(I^\omega, Homeo(I^\omega))$ we get $\mathcal{NM}(I^\omega) = 1$.
5. $(P, Homeo(P))$ is small but not nm-stable.

10. Polish G-groups

Definition A Polish [compact] G-group is a Polish structure (H, G) such that G acts continuously and by automorphisms on the Polish [compact] group H.

Examples:

1. Profinite groups regarded as profinite structures.

2. $(H, Aut(H))$ where H is a compact metric group and $Aut(H)$ is the group of all topological automorphisms of H equipped with the compact-open topology.

Example of a small Polish, non-compact G-group:

$H := (\mathbb{Q}^\omega, +)$ with the product topology where $\mathbb{Q}$ is equipped with the discrete topology.

$G := Aut^0(\mathbb{Q}^\omega)$ is the group of all automorphisms of $\mathbb{Q}^\omega$ preserving the inverse system $\mathbb{Q} \longleftarrow \mathbb{Q} \times \mathbb{Q} \longleftarrow \dots$.

Since

$$G = \varprojlim Aut^0(\mathbb{Q}^n),$$

we can introduce the inverse limit topology on G, putting the pointwise convergence topology on each $Aut^0(\mathbb{Q}^n)$.

Then (H, G) is a small, Polish, non-compact G-group of $\mathcal{NM}$-rank 1. It is also 0-dimensional and torsion free.

Problem Find non 0-dimensional small Polish G-groups.

11. Small compact G-groups

From now on, (H, G) is a **small compact G-group**.

Remark H is locally finite.

Fact (Hewitt, Ross) Every compact torsion group is profinite.

Corollary H is profinite; in fact, it is the inverse limit of a countable system.

The difference between our situation and profinite groups regarded as profinite structures is that G is not necessarily compact (profinite).

Conjecture (Newelski) Each small profinite group is abelian-by-finite.

Theorem (Wagner) Each small, nm-stable profinite group is abelian-by-finite.

Intermediate conjectures: Suppose (H, G) is a small profinite group. Then:

(A) H is solvable-by-finite;

(B) if H is solvable(-by-finite), then it is nilpotent-by-finite;

(C) if H is nilpotent(-by-finite), then it is abelian-by-finite.

Proposition (K) In our more general context where (H, G) is a small compact G-group, all conjectures (A), (B) and (C) are false.

Counter-example for (A):

S – any finite, non-solvable group

S_∞ acts on S^ω by

$$g\langle s_0, s_1, \ldots \rangle = \langle s_{g(0)}, s_{g(1)}, \ldots \rangle.$$

Then:

(i) S^ω is not solvable-by-finite,

(ii) (S^ω, S_∞) is a small compact S_∞-group,

(iii) (S^ω, S_∞) is not nm-stable.

Theorem (K) If (H, G) is a small, nm-stable, compact G-group, then H is solvable-by-finite.

The proof uses Wilson's theorem on the structure of profinite torsion groups, Frattini subgroups, basic topology and forking calculus.

Theorem (K) If (H, G) is a small compact G-group of finite $\mathcal{NM}$-rank, and H is solvable-by-finite, then H is nilpotent-by-finite.

The proof is based on Newelski's proof of the corresponding result for small profinite groups. However, the proof in our context is more complicated.

Corollary (K) If (H, G) is a small compact G-group of finite $\mathcal{NM}$-rank, then H is nilpotent-by-finite.

Conjecture Each small, nm-stable compact G-group is abelian-by-finite.

Proposition (K) Each small, compact G-group of $\mathcal{NM}$-rank 1 is abelian-by-finite.

For Polish G-groups everything is more complicated.

Question Is it true that every small, nm-stable Polish G-group is abelian-by-countable?

Even the following question is open.

Question Is every small, Polish G-group of $\mathcal{NM}$-rank 1 abelian-by-countable?

12. Future investigations of Polish structures

1. Prove counterparts of some deep model theoretic results.

2. Find further examples, especially of small Polish G-groups; understand which compact metric spaces with the full group of homeomorphisms are small.

3. Investigate the structure of groups and rings.

4. Try to find counterexamples for some difficult conjectures about small profinite structures in the wider context of small Polish structures.

5. Try to apply the introduced model theoretic tools to purely descriptive set theoretic problems.

6. Try to find a notion of interpretability in first order structures.