Properties of PA sets and random sets

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The main result, a joint work with T. Slaman

- There is a low T-upper bound for the class of K-trivials
- A characterization of ideals in Δ^0_2 degrees which have a low T-upper bound

Definition

Let $\mathcal{PA}(B)$ denote the class of $\{0,1\}$-valued B-DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x .
If B is $\emptyset$ we simply speak of $\mathcal{PA}$.

Definition

Let $\mathcal{DNR}(B)$ denote the class of B-DNR functions, i.e. the class of functions $f \in \omega^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x.
If B is $\emptyset$ we simply speak of $\mathcal{DNR}$.

Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Theorem (D. Scott and others)

The following conditions are equivalent:

1. $\mathbf{a}$ *is a degree of a* $\{0,1\}$*-DNR function*
2. $\mathbf{a} >> \mathbf{0}$
3. $\mathbf{a}$ *is a degree of a complete extension of PA*
4. $\mathbf{a}$ *is a degree of a set separating some effectively inseparable pair of r.e. sets.*

Remark

1. $\mathcal{PA}$ is a kind of a "universal" Π^0_1 class
2. $\{0,1\}$-valued DNR functions are also called PA sets and degrees $>> \mathbf{0}$ are called PA degrees.
3. (Simpson)
   (a) The partial ordering $<<$ is dense
   (b) $\mathbf{a} << \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$.

Known facts

The class of PA degrees is closed upwards (it forms an upper cone). The class of sets which have a PA degree has measure 0.

Remark

The first part gives an example of coding into PA sets, based on Gödel incompleteness phenomenon.
(More on that later).

Definition

Let M be an infinite set and $\{m_0, m_1, m_2, \ldots\}$ be an increasing list of all members of M.

- If $f \in 2^\omega$ then by $Restr(f, M)$ we denote $g \in 2^\omega$ defined for all i by $g(i) = f(m_i)$
- Similarly, if $\mathcal{A} \subseteq 2^\omega$ then by $Restr(\mathcal{A}, M)$ we denote a class of functions $\{g : g = Restr(f, M) \ \wedge \ f \in \mathcal{A}\}$.

(*Idea: an analogue of a projection.*)

Lemma (A.K.)

- *For every Π^0_1 class $\mathcal{A} \subseteq \mathcal{PA}$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, i.e. for every $g \in 2^\omega$ there is a function $f \in \mathcal{A}$ such that $Restr(f, M) = g$.*
- *For every $\Pi^{0,B}_1$ class $\mathcal{A} \subseteq \mathcal{PA}(B)$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, where (an index of) M can be found uniformly from an index of $\mathcal{A}$, i.e. it does not depend on B.*

Remark

- This is basically Gödel incompleteness phenomenon
- It can be modified to a dynamic process, i.e. given an effective sequence of Σ^0_1 and Π^0_1 events, we can close (i.e. code) true Σ^0_1 ones while leaving open true Π^0_1 ones.

The Lemma is crucial for coding into members of (nonempty) Π^0_1 classes $\mathcal{A}$ which are subclasses of $\mathcal{PA}$.

We may

- code either an individual set C (by $Restr(\mathcal{A}, M) = \{C\}$)
- or nest another class $\mathcal{E} \subseteq 2^\omega$ (by $Restr(\mathcal{A}, M) = \mathcal{E}$)

Similarly with coding into members of nonempty $\Pi^{0,B}_1$ classes which are subclasses of $\mathcal{PA}(B)$.

Nesting in this way a $\Pi^{0,C}_1$ class into a $\Pi^{0,B}_1$ class we obtain $\Pi^{0,B\oplus C}_1$ class.

Example

Z is a low set then there is a low PA set A such that $Z \leq_T A$.

Algorithmic randomness

K denotes prefix-free Kolmogorov complexity
$\{\mathcal{U}_n : n \in \omega\}$ denotes a universal ML test
1-randomness (ML-randomness) and relativization

Schnorr (equivalent characterizations of 1-randomness):

For any set A, $K(A \upharpoonright n) \geq n + O(1)$, if and only if
A passes all ML-tests (equivalently, $A \notin \bigcap_n \mathcal{U}_n$)

1-random sets

- form a Σ^0_2 class of measure 1
- $= \{\sigma * A \ : \ A \notin \mathcal{U}_n \ \& \ \sigma \in 2^{<\omega}\}$ (any n)

Thus, up to a finite shift, 1-random sets are just members of a Π^0_1 class (like $\overline{\mathcal{U}_n}$).

We work with Π^0_1 classes of positive measure (a kind of thick Π^0_1 classes) which are in a sense universal for Π^0_1 classes of positive measure.

From any 1-random set it is possible to compute a DNR function 1-randomness is a special case of a diagonalization of some Σ^0_1 objects (effective approximations in measure).

Algorithmic weakness

There are several notions of computational weakness related to 1-randomness

Definition

1. $\mathcal{L}$ denotes the class of sets which are low for 1-randomness, i.e. sets A such that every 1-random set is also 1-random relative to A.
2. $\mathcal{K}$ denotes the class of K-trivial sets, i.e. the class of sets A such that for all n, $\quad K(A \upharpoonright n) \leq K(0^n) + O(1)$.
3. $\mathcal{M}$ denotes the class of sets that are low for K, i.e. sets A such that for all σ, $\quad K(\sigma) \leq K^A(\sigma) + O(1)$.
4. A set A is a basis for 1-randomness if $A \leq_T Z$ for some Z such that Z is 1-random relative to A. The collection of such sets is denoted by $\mathcal{B}$.

Theorem (Nies, Hirschfeldt, Stephan)

$\mathcal{K} = \mathcal{L} = \mathcal{M} = \mathcal{B}$

More precisely:

- Nies: $\mathcal{L} = \mathcal{M}$
- Hirschfeldt, Nies: $\mathcal{K} = \mathcal{M}$
- Hirschfeldt, Nies, Stephan: $\mathcal{K} = \mathcal{B}$

Four different characterizations of the same class!
However, these characterizations yield different information content

Basic facts about $\mathcal{K}$

- $\mathcal{K} \subseteq \Delta^0_2$
- $\mathcal{K} \subseteq L_1$ (i.e. K-trivials are low)

More precisely:

- Chaitin: $\mathcal{K} \subseteq \Delta^0_2$
- A.K.: $\mathcal{L} \subseteq GL_1$ (thus, $\mathcal{L} = \mathcal{K} \subseteq L_1$)

Nowadays there are easier ways to prove lowness of K-trivials

Theorem (Nies; Downey, Hirschfeldt, Nies, Stephan)

- *r.e. K-trivial sets induce a Σ^0_3 ideal in the r.e. T-degrees*
- *K-trivial sets induce an ideal in the ω-r.e. T-degrees generated by its r.e. members (in fact, a Σ^0_3 ideal in the ω-r.e. T-degrees)*

Theorem (Downey, Hirschfeldt, Nies, Stephan; Nies)

- *There is an effective sequence* $\{B_e, d_e\}_e$ *of all the r.e. K-trivial sets and of constants such that each B_e is K-trivial via d_e*
- *There is no effective sequence* $\{B_e, c_e\}_e$ *of all the r.e. low for K sets with appropriate constants*
- *There is no effective way to obtain from a pair (B, d), where B is an r.e. set that is K-trivial via d, a constant c such that B is low for K via c*
- *There is no effective listing of all the r.e. K-trivial sets together with their low indices*

Theorem (Nies)

For each low r.e. set B, there is an r.e. K-trivial set A such that $A \not\leq_T B$.

Thus, no low r.e. set can be a T-upper bound for the class $\mathcal{K}$.

Comment

The proof uses Robinson low guessing technique which is compatible for r.e. sets with a technique do what is cheap. Cheap

is defined

- either by a cost function in case of K-trivials,
- or by having a small measure in case of low for random sets.

However, in the more general case of Δ^0_2 instead of r.e. sets, the Robinson low guessing technique does not seem to be compatible with a technique do what is cheap. In fact, it is not.

Since all K-trivials are low and every K-trivial set is recursive in some r.e. K-trivial set, we have, as a corollary, that the ideal (induced by) $\mathcal{K}$ is nonprincipal (in the Δ_2^0 T-degrees)

A more general result.

Theorem (Nies)

For any effective listing $\{B_e, z_e\}_e$ of low r.e. sets and of their low indices there is an r.e. K-trivial set A such that $A \not\leq_T B_e$ for all e.

This result is, in fact, used to prove that there is no effective way to obtain low indices of (r.e.) K-trivial sets

Theorem (Nies)

- *There is a low$_2$ r.e. set which is a T-upper bound for the class of K-trivials.*
- *Any proper Σ^0_3 ideal in the r.e. T-degrees has a low$_2$ r.e. T-upper bound*

Question

Is there a low Δ^0_2 T-upper bound for the class $\mathcal{K}$?

Theorem (Yates)

For any r.e. set A TFAE:

1. $A'' \equiv_T \emptyset''$
2. $\{x : W_x \leq_T A\}$ *is a* Σ^0_3 *set*
3. *the class* $\{W_x : W_x \leq_T A\}$ *is uniformly r.e.*

Together with Nies' result, we have the following characterization.

Fact

An ideal of r.e sets has a low_2 r.e. T-upper bound if and only if it is a subideal of a proper Σ^0_3 ideal.

Open

A characterization of Σ^0_3 ideals in the r.e. T-degrees for which there is a low T-upper bound, not necessarily r.e.(!)
(similarly for ideals in Δ^0_2 T-degrees)

Theorem (A.K., Slaman)

Let $\mathcal{C}$ be a Σ^0_3 ideal in the r.e. T-degrees. Then TFAE:

1. *there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low T-upper bound for $\mathcal{C}$*

A slightly more general result.

Theorem (A.K., Slaman)

Let $\mathcal{C}$ be an ideal in Δ^0_2 T-degrees. Then TFAE:

1. *(a) $\mathcal{C}$ is contained in an ideal $\mathcal{A}$ which is generated by a sequence of sets $\{A_n\}_n$ such that the sequence is uniformly recursive in $\emptyset'$ and*
   (b) there is a function F recursive in $\emptyset'$ which dominates any partial function recursive in any set with T-degree in $\mathcal{A}$,
2. *there is a low T-upper bound for $\mathcal{C}$.*

Corollary

There is a low T-upper bound for the class $\mathcal{K}$ (the class of K-trivials).

Proof

Nies proved that the ideal (induced by) $\mathcal{K}$ is generated by its r.e. members and r.e. K-trivial sets induce a Σ_3^0 ideal in the r.e. T-degrees.
A.K. and Terwijn proved that there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of $\mathcal{K}$ $\{$Remark: Jump traceability of K-trivials is implicit in this result$\}$.
Thus, Corollary follows from the previous Theorem.

Remark

Since every low set has a low PA set T-above it, low T-upper bounds which are PA are the most general case in this characterization.

The following lemma is the heart of the matter.

Lemma

Given a function F recursive in $\emptyset'$, there is a uniform way how to obtain from a $\emptyset'$-index of a set A with the property that any partial function recursive in A is dominated by F both a low set A^* and an index of lowness of A^* such that $A \leq_T A^*$,
i.e. there are recursive functions f, g such that if $\Phi_e(\emptyset')$ is total and equal to some set A so that any partial function recursive in A is dominated by F then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $A \leq_T \Phi_{f(e)}(\emptyset')$.

Comment.

In general, it is not possible to reach $A \leq_T A^*$ uniformly in an index of A, otherwise we would have a contradiction with a result of Nies (no effective listing of K-trivials together with their low indices).
Similarly, sets A^* cannot be, in general, obtained uniformly as r.e. sets.

Main idea

To combine forcing with Π^0_1 classes (like Low Basis Theorem) with coding sets into rich Π^0_1 classes, namely into subclasses of $\mathcal{PA}$.

Idea of the proof of the main lemma (given a function F recursive in $\emptyset'$ and a set A with described properties).

An extremely simplified version : having a low index of A.

1. Code A into $\mathcal{PA}$, and get a $\Pi_1^{0,A}$ class by $Restr(\mathcal{PA}, M) = \{A\}$, where M is an infinite recursive set used for coding,
   or more generally, by $Restr(\mathcal{PA}, M) = \mathcal{PA}(A)$
   (here we may repeat nesting, i.e. coding into $\mathcal{PA}(A)$)
2. Apply relativized Low Basis Theorem to get a member of the class.

A full version: we do not have a low index of A.

Missing low index of A is replaced by approximations provided by F to A'-questions. Since $(A^*)'$ has to be uniformly recursive in $\emptyset'$, our $\emptyset'$-construction of both A^* and $(A^*)'$ cannot change any decision about $(A^*)'(x)$ that it has already made. A wrong approximation to A'-question given by F leads eventually to a conflict with coding of A. We have to keep all our commitments about $(A^*)'(x)$ that we have already made and we have to start with a new coding strategy.
If A and F satisfy the given assumptions our method will guarantee that the approximations given by F will be correct from some point on, i.e. a coding strategy will eventually stabilize yielding $A \leq_T A^*$. Since we use Π^0_1 subclasses of $\mathcal{PA}$, we can always find a place for a new coding strategy (i.e. for coding an infinitary information). Here we substantially use the fact that (nonempty)

Π^0_1 subclasses of $\mathcal{PA}$ are rich

We use terms:
ω-extendability and F-extendability of a string in a tree (consider recursive trees yielding Π^0_1 classes $\subseteq \mathcal{PA}$ or A-recursive trees yielding $\Pi^{0,A}_1$ classes $\subseteq \mathcal{PA}(A)$).

Shortly, strings may be ω-good, $F(...)$-good etc.

We always have to keep

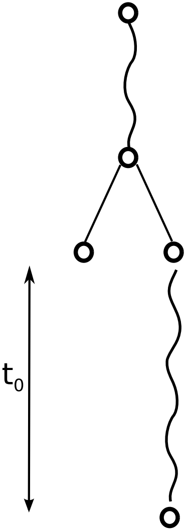
- ω-extendability of our strings in our recursive trees (trees for Π^0_1 subclasses of $\mathcal{PA}$)
- (only) F-extendability of these strings in A-recursive trees (trees for $\Pi^{0,A}_1$ subclasses of $\mathcal{PA}(A)$).

We explain the idea on a picture (first some notation).

Let G be a recursive function such that

$\lim_s G(\alpha, s) = F(\alpha)$.

We build an A-partial recursive function H, such that whenever we are in a real trouble, the value of H at such place will be greater than the value of F.
Since F has to eventually dominate H, from some point on there is no trouble at all and we win, i.e. a coding strategy will be stable and F-extendability will be, in fact, ω-extendability.



F-good, ω-bad (F doesn't know !)
in an A-recursive tree

still F-good, ω-bad (F doesn't know !)
$d_\emptyset$ = extendability, $H(..) = d_\emptyset > F(..)$

both F-bad, d_0, d_1 = extenadability
$F(..) > d_0, d_1$
F knows ! but A doesn't know !

Wait for t_0 with $G(.., t_0) > d_0, d_1$
Here A knows !
We can synchronize $\emptyset'$ and A-construction
We start a new coding strategy here
Note: a finite injury is behind, A doesn't know efectively where this happens

This explains how to prove Lemma, i.e. how to deal with just one fixed A_n.

To prove the main Theorem, i.e. to deal with all given sets A_n we have to:
first, relativize Lemma and work with Π_1^{0,A_n} classes and,
second, subsequently nest all Π_1^{0,A_n} classes, i.e. at each step we nest subsequent Π_1^{0,A_n} class into a previous one.

At each level of nesting there are only finitely many injuries and our construction eventually reaches all goals.

As a corollary of a result of Nerode and Shore there is an exact pair for the class $\mathcal{K}$ in Δ_2^0 T-degrees.

Question

1. Is there an exact pair for the class $\mathcal{K}$ in the r.e. T-degrees?
2. Is there a low exact pair for the class $\mathcal{K}$ in the Δ_2^0 T-degrees?

Comment

The desribed method (to produce a low T-upper bound) does not seem to be easily applicable to produce low exact pairs for ideals in question.
Example: there is no minimal pair of PA degrees below $\mathbf{0}'$.

Coding into 1-random sets

On the contrary to (Π^0_1 classes of) PA sets, where we have

a coding place (coding bit)

for (Π^0_1 classes of) 1-random sets we only have

a coding interval (Kučera-Gács coding).

Examples

- Any set A is T-reducible to a 1-random set Z (even *wtt*) ($A \leq_{wtt} Z \leq_T A \oplus \emptyset'$), and
- $\{\mathbf{a} : \mathbf{a} \geq \mathbf{0}'\} \subseteq$ 1-random degrees.

R.sp., complexity of coding (into 1-random sets) is $\mathbf{0}'$. It agrees with our intuition: it is not possible to arbitrarily code an infinite information into 1-random objects keeping 1-randomness (chaoticness).

1-random degrees are not closed upwards, or strongly :

Fact (Stephan)

1-random degrees an PA degrees coincide exactly on $\{\mathbf{a} : \mathbf{a} \ \geq \ \mathbf{0}'\}$.

In a connection with low T-upper bounds for K-trivials there is a very difficult and sharp question.

Question

Is there a low 1-random set which is a T-upper bound for the class $\mathcal{K}$?

There is a very strong limitation on coding an r.e. set into incomplete 1-random sets.

Theorem (Hirschfeldt, Nies, Stephan)

If B is r.e. and Z is 1-random such that $B \leq_T Z$ and $Z \not\geq_T \emptyset'$ then Z is 1-random in B and, thus, B is K-trivial.

Thus, the only r.e. sets which may have incomplete 1-random set T-above are K-trivials. (At least some nonrecursive of them do).

Corollary

Suppose A is both 1-random and a low T-upper bound for the class $\mathcal{K}$. Then $\quad$ *r.e. K-trivials* $= \{B : B \text{ is r.e. } \& \ B \leq_T A\}$.

So, an existence of a low 1-random T-upper bound for the class $\mathcal{K}$ would be a very strong result.

The mentioned limitation also indicates that to code an r.e. K-trivial set into an incomplete 1-random set requires to use some global dynamic property connected with K-triviality of a given set (rather than to code an r.e. set bit by bit).

A weaker variant of the question (for just one fixed K-trivial).

Question

Does for any (r.e.) K-trivial set A exist an incomplete (e.g. low) 1-random set Y such that $A \leq_T Y$?

Apology

It is announced in my abstract that Barmpalias and Montalban proved that any K-trivial set is T-below some low 1-random set. Unfortunately, they found recently a gap in the construction.

The method of coding they used is interesting.

Theorem (Nies, Stephan; Kjos-Hanssen)

A set A is K-trivial if and only if for any $\Sigma_1^{0,A}$ class $\mathcal{U}^A$ of measure < 1 there is a Σ_1^0 class $\mathcal{V}$ of measure < 1 such that $\mathcal{U}^A \subseteq \mathcal{V}$.

Definition

$A \leq_{LR} B$ if every set 1-random in B is also 1-random in A, i.e. $MLR^B \subseteq MLR^A$.

Observe: $\mathcal{K} = \mathcal{L} = \{A : A \leq_{LR} \emptyset\}$.

Theorem (Kjos-Hanssen)

$A \leq_{LR} B$ if and only if for any $\Sigma_1^{0,A}$ class $\mathcal{U}^A$ of measure < 1 there is a $\Sigma_1^{0,B}$ class $\mathcal{V}^B$ of measure < 1 such that $\mathcal{U}^A \subseteq \mathcal{V}^B$.

This characterization gives some "global dynamic property" of K-trivials. R.sp., having $\mathcal{U}^A \subseteq \mathcal{V}$, with measure of $\mathcal{V} < 1$, $\mathcal{V}$ can be used as a test for a confirmation that some σ is an initial segment of A. This global bound $\mathcal{V}$ of measure < 1 can guarantee that we can keep the measure of mistakes small.

The idea of Barmpalias and Montalban (to construct a low 1-random set T-above a given K-trivial set A) was based on this, namely, $\mathcal{U}^A \subseteq \mathcal{V}$,
which enables to correct mistakes in approximations to A with measure of mistakes small. It could eventually produce a T-reduction of A to some low 1-random set. Unfortunately, there is gap in the current version and it is not clear whether a much more nonuniform version could work. So, it is open.

Opposite to lowness: highness.

Definition

$LRH \;=\; \{A:\; \emptyset' \leq_{LR} A\}$ (LR-hard)

Definition (Diamond operator)

For a class $\mathcal{H} \subseteq 2^{\omega}$ let

$\mathcal{H}^{\diamond} \;=\; \{A:\; A \; r.e. \; \& \; \forall Z \in \mathcal{H} \cap MLR (A \leq_T Z)\}$

Obviously, $\mathcal{H}^{\diamond}$ induces an ideal in the r.e. T-degrees.
There are several subclasses of the r.e. T-degrees of the form $\mathcal{H}^{\diamond}$ (more on that in Nies' book: Computability and randomness).

Example

For $\mathcal{H} = LHR$ we have

$LRH^{\diamond} \;=\; \{A:\; A \; r.e. \; \& \; \forall Z \in LRH \cap MLR (A \leq_T Z)\}$

Theorem (Nies)

There is a 1*-random set* $A \in LRH$ *such that* $A <_T \emptyset'$*.*

Remark

Alternatively, the jump inversion technique for Π^0_1 classes (A.K. 1989: high incomplete 1-random) yields immediately also a pseudo-jump inversion method and, thus, also produces a 1-random set $A \in LRH$ such that $A <_T \emptyset'$.

See: Simpson `http://www.math.psu.edu/~simpson`
paper: Mass Problems and Almost Everywhere Domination

(Other paper of Simpson is about LR-reducibility, almost everywhere domination, a relation $\emptyset' \leq_{LR} A$, etc. is: Almost Everywhere Domination and Superhighness.)

Corollary

$LRH^\diamond \subseteq \mathcal{K}$ (in fact, $LRH^\diamond \subseteq$ r.e. members of $\mathcal{K}$)

Theorem (Hirschfeldt, Miller)

For every Σ^0_3 null class $\mathcal{C}$ there is a nonrecursive r.e. set which is T-below all 1-random sets in $\mathcal{C}$.

Corollary (Hirschfeldt, Miller)

$LRH^\diamond$ (a subclass of the r.e. members of $\mathcal{K}$) contains also nonrecursive r.e. sets.

LRH sets in $\mathcal{PA}$

Theorem (A.K.)

- *There is a PA set A, $A <_T \emptyset'$, $A \in LRH$, (i.e. $\emptyset' \leq_{LR} A$)*
- *For every nonrecursive $Z \leq_T \emptyset'$, there is a PA set $A \in LRH$ such that $A <_T \emptyset'$ & $A \oplus Z \equiv_T \emptyset'$.*

(2^{nd} item: Posner-Robinson by incomplete PA sets from LRH)
Thus, the only sets T-below all $LRH \cap \mathcal{PA}$ are computable.

Another interesting contrast of 1-random and PA sets:

Fact (Hirschfeldt)

Sets in $LRH^\diamond$ are ML-noncuppable, i.e.
$A \in LRH^\diamond \;\rightarrow\; A \oplus Z <_T \emptyset'$ for all 1-random set $Z <_T \emptyset'$.

Observe: there is no cone avoidance by 1-random members in *LRH* (at least some nonrecursive r.e. K-trivials are T-below all 1-random *LRH* sets)

on the contrary to the case of (incomplete) PA sets in *LRH* (we even have a variant of Posner-Robinson by incomplete PA sets in *LRH*).

Question

- $LHR^{\diamond} =$ r.e. members of $\mathcal{K}$?
- Are all K-trivials ML-noncuppable ?

Comment

Many obstacles in solving the above questions concerning 1-randomness are connected with a problem of coding an information into 1-random sets.

While we can code an infinitary information into *PA* sets (or into members of Π^0_1 subclasses of $\mathcal{PA}$),

coding an information into 1-random sets
(or into members of Π^0_1 classes of positive measure)
is less powerful and it is still not completely understood.

The paper about low T-upper bounds of ideals is submitted to a journal, a preprint can be found at

`http://math.berkeley.edu/~slaman/papers`
(a revised version will be available soon).

Thank you