Axiomatic Theories of Truth

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Formalising Truth

Definition (Language of truth)

We work in $\mathcal{L}_T$ the language of Peano Arithmetic augmented with an additional predicate symbol T. Let PA_T denote PA formulated in the language $\mathcal{L}_T$.

The intuition is that $T(x)$ denotes that x is (the Gödel number of) a "true" $\mathcal{L}_T$ sentence.

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Choices for truth

Of course, by Tarski's Theorem the "ideal" axiom of truth, $T\ulcorner A \urcorner \leftrightarrow A$ for all sentences A, is inconsistent with PA_T. However, there are ways in which we can overcome this inconsistency.

1. Restrict the language so as to stop self-reference. For example allow $T\ulcorner A \urcorner \leftrightarrow A$ for $\mathcal{L}_{PA}$.
2. Replace $T\ulcorner A \urcorner \leftrightarrow A$ with weaker, consistent, axioms.

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A Base for truth

Let $Base_T$ be the theory comprising of PA_T and the following axioms.

1. $(T\ulcorner A \rightarrow B\urcorner \wedge T\ulcorner A\urcorner) \rightarrow T\ulcorner B\urcorner$.
2. $T(\mathbf{ucl}\ulcorner B\urcorner)$ for all tautologies B.
3. $T\ulcorner A\urcorner$ if A is a true primitive recursive atomic sentence.

where $\mathbf{ucl}(A)$ denotes the (Gödel number of the) universal closure of A.

Axioms of truth

Possible axioms, schema, and rules of inference we consider are

$$
\begin{array}{lll}
A \rightarrow T\ulcorner A\urcorner, & \neg(T\ulcorner A\urcorner \wedge T\ulcorner \neg A\urcorner), & A/T\ulcorner A\urcorner, \\
T\ulcorner A\urcorner \rightarrow A, & T\ulcorner A\urcorner \vee T\ulcorner \neg A\urcorner, & T\ulcorner A\urcorner/A, \\
T\ulcorner A\urcorner \rightarrow T\ulcorner T\ulcorner A\urcorner\urcorner, & \forall n\, T\ulcorner A\dot{n}\urcorner \rightarrow T\ulcorner \forall x\, Ax\urcorner, & \neg A/\neg T\ulcorner A\urcorner, \\
T\ulcorner T\ulcorner A\urcorner\urcorner \rightarrow T\ulcorner A\urcorner, & T\ulcorner \exists x\, Ax\urcorner \rightarrow \exists n\, T\ulcorner A\dot{n}\urcorner, & \neg T\ulcorner A\urcorner/\neg A.
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These axioms were considered by Harvey Freidman and Michael Sheard in *An axiomatic approach to self-referential truth* [2].

They classified the above axioms and rules into nine maximally consistent sets.

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U-Inf: $\forall n\, T\ulcorner A\dot{n}\urcorner \rightarrow T\ulcorner \forall x\, Ax \urcorner$ T-Elim: $T\ulcorner A\urcorner / A$
T-Del: $T\ulcorner T\ulcorner A\urcorner\urcorner \rightarrow T\ulcorner A\urcorner$ T-Intro: $A / T\ulcorner A\urcorner$

Definition

Let S_1 be $Base_T$ + U-Inf + T-Elim, and
S_2 be $Base_T$ + U-Inf + T-Del + T-Intro + T-Elim.
Denote by $I(\alpha)$ the formula $\forall\ulcorner A\urcorner\, T\ulcorner \mathrm{TI}(\dot{\alpha}, \dot{A})\urcorner$.

Sheard proved (in [4]) that $S_1 \vdash \forall\alpha.\ I(\alpha) \rightarrow I(\epsilon_\alpha)$. Moreover he showed $|S_1| = \varphi 20$.
We have shown $S_2 \vdash \forall\alpha.\ I(\alpha) \rightarrow I(\varphi n \alpha)$ for each n.

Proof. (Sketch).

From $\vdash \forall\alpha.\ I(\alpha) \rightarrow I(\varphi n\alpha)$ we get $\vdash \mathrm{Prog}_\beta I(\varphi n'\beta)$ (with U-Inf). Thus, $\vdash \forall\alpha.\ \mathrm{TI}_\beta(\alpha, I(\varphi n'\beta)) \rightarrow I(\varphi n'\alpha)$. Now by T-Intro, axioms of $Base_T$ and T-Del we have $\vdash \forall\alpha.\ T\ulcorner \mathrm{TI}_\beta(\alpha, I(\varphi n'\beta))\urcorner \rightarrow I(\varphi n'\alpha)$. □

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It is more interesting to find an upper bound for S_2. For this we need to take a detour into infinitary logic.

Definition (Inductive Definition of S_2^∞)

Define $S_2^\infty \overset{\alpha,n}{\underset{k}{\vdash}} \Gamma$ by (Ax.1), ($\wedge$), ($\vee_i$), (ω), ($\exists$) and

(Cut). If $\overset{\alpha,n}{\underset{k}{\vdash}} \Gamma, A$, $\overset{\delta,n}{\underset{k}{\vdash}} \Gamma, \neg A$ and $|A| < k$ then $\overset{\beta,n}{\underset{k}{\vdash}} \Gamma$,

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if $\alpha, \delta < \beta$.

Definition

The rank of A, $|A|$, is defined as follows.

- $|A| = 0$ if A is an arithmetical literal or $T(s)$ for some term s.
- $|A \wedge B| = |A \vee B| = |\forall x\, A| = |\exists x\, A| = |A| + 1$.

Theorem

Cut Elimination

$$S_2^\infty \vdash^{\alpha, n}_{k+1} \Gamma \ \textit{ implies } \ S_2^\infty \vdash^{\omega^\alpha, n}_{k} \Gamma\,.$$

For each $n > 0$ and α define

$$\mathfrak{M}_{n,\,\alpha} = \left\langle \mathbf{N}, \{\ulcorner B \urcorner : S_2^\infty \big|\!\frac{\alpha_0, m}{0}\, B \ \text{ for some } m < n \text{ and } \alpha_0 < \alpha\} \right\rangle$$

and define $\mathfrak{M}_{0,\,\alpha} = \langle \mathbf{N}, \emptyset \rangle$.

Lemma

For each n define $f_n(\alpha) = \varphi n(\varphi 1 \alpha)$. Then for every $n < \omega$ we have

1. *If* $\big|\!\frac{\alpha,\, n}{0}\, \Gamma$ *then* $\mathfrak{M}_{n,\, f_n(\alpha)} \models \Gamma$.
2. *If* $\big|\!\frac{\alpha,\, n}{0}\, T\ulcorner A \urcorner$ *then* $\big|\!\frac{f_n(\alpha),\, p}{0}\, A$ *for some* $p < n$.
3. *If* $\big|\!\frac{\alpha,\, n}{k}\, \Gamma$ *then* $\big|\!\frac{\varphi 1 \alpha,\, n}{0}\, \Gamma$.

Corollary

If $\alpha < \varphi\omega 0$ *then* $\big|\!\frac{\alpha, n}{0}\, T\ulcorner A \urcorner$ *implies* $\big|\!\frac{\beta, n}{0}\, A$ *for some* $\beta < \varphi\omega 0$.

For each $n > 0$ and α define

$$\mathfrak{M}_{n,\alpha} = \left\langle \mathbf{N}, \{\ulcorner B \urcorner : S_2^\infty \mid\!\frac{\alpha_0, m}{0}\, B \text{ for some } m < n \text{ and } \alpha_0 < \alpha\} \right\rangle$$

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and define $\mathfrak{M}_{0,\,\alpha} = \langle \mathbf{N}, \emptyset \rangle$.

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For each n define $f_n(\alpha) = \varphi n(\varphi 1 \alpha)$. Then for every $n < \omega$ we have

1. *If $\vdash_0^{\alpha,\,n} \Gamma$ then $\mathfrak{M}_{n,\, f_n(\alpha)} \models \Gamma$.*
2. *If $\vdash_0^{\alpha,\,n} T\ulcorner A \urcorner$ then $\vdash_0^{f_n(\alpha),\,p} A$ for some $p < n$.*
3. *If $\vdash_k^{\alpha,\,n} \Gamma$ then $\vdash_0^{\varphi 1 \alpha,\,n} \Gamma$.*

Corollary

If $\alpha < \varphi\omega 0$ then $\vdash_0^{\alpha,n} T\ulcorner A \urcorner$ implies $\vdash_0^{\beta,n} A$ for some $\beta < \varphi\omega 0$.

Thus, we have

Lemma

If $S_2 \vdash A$ then $S_2^\infty \vdash^{\alpha,n}_{0} A$ for some $\alpha < \varphi\omega 0$.

and

Theorem

Let A be an arithmetical sentence, then $S_2 \vdash A$ implies $PA + \mathsf{TI}(<\varphi\omega 0) \vdash A$.

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Corollary (T-Elimination for S_2^∞)

$|S_2| = \varphi\omega 0$.

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U-Inf: $\forall n\, T\ulcorner A\dot{n}\urcorner \to T\ulcorner \forall x\, Ax\urcorner$, T-Elim: $T\ulcorner A\urcorner / A$, T-Cons:
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The bounds for S_2 were fairly easy to establish. However, this is not the case for all nine of the theories we considered.

For example, $\mathcal{E}$ is given by

$$Base_T + \text{T-Del} + \text{U-Inf} + \text{T-Cons} + \text{T-Intro} + \text{T-Elim}.$$

The upper bound $\mathcal{E}$ is not so clear because we no longer have Cut-Elimination in the corresponding infinitary system. However, we can embed $\mathcal{E}$ in a small extension of ID_1^*.

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ID_1^* is the theory extending PRA in which for each arithmetical formula $A \in \mathcal{L}_P^+$ with only one free variable the language is augmented by an additional predicate symbol I_A and we have the axioms

$$\forall u.\ A(u, I_A) \to I_A(u), \tag{Ax.I_A.1}$$

$$\forall u[A(u, F) \to F(u)] \to \forall u[I_A(u) \to F(u)], \tag{Ax.I_A.2}$$

for each formula F containing only positive occurrences of predicates I_B for $B \in \mathcal{L}_P^+$ and induction for formulae where fixed-point predicates occur positively.

We define ID_1^{*+} to be ID_1^* with, as an additional axiom,

$$\forall u[A(u, F) \to F(u)] \to \forall u[I_A(u) \to F(u)], \tag{Ax.I_A.3}$$

if $A \in \mathcal{L}_P^+$ is Σ_2 and F is any formula which is Σ_1 or Π_1 in I_A and $\neg I_A$

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Theorem

There are formulae $A_n(u, P^+)$ such that $\mathcal{E} \vdash C$ implies there is an n such that $\mathrm{ID}_1^{*+} \vdash I_n(\ulcorner C^* \urcorner)$.

Theorem

Every arithmetical consequence of $\mathcal{E}$ is a theorem of ID_1^{*+}.

Proof.

Let $\mathfrak{A}$ be a model for the first-order part of ID_1^{*+}. Using $\mathfrak{A}$ we may then build a hierarchy of $\mathcal{L}_T$-structures

$$\mathfrak{M}_0 = \langle \mathfrak{A}, \emptyset \rangle \,;$$
$$\mathfrak{M}_{n+1} = \langle \mathfrak{A}, I_n \rangle \,.$$

with the property that $\mathfrak{M}_n \models I_n$. Now, if $\mathcal{E} \vdash A$ then $\mathfrak{M}_n \models A$ for some n. Hence $\mathfrak{A} \models A$. But $\mathfrak{A}$ was arbitrary. $\square$

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And so,

Theorem

$\varphi\omega 0 = |\mathrm{ID}_1^*| \leq |\mathcal{E}| \leq |\mathrm{ID}_1^+|$.

References

Thank you.

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