INTERPOLATION AND RELATED PROPERTIES IN SEMILATTICE BASED VARIETIES

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Outline

1. Abstract
2. Interpolation and amalgamation
3. Semilattice based varieties
4. Varieties associated with non-classical logics

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ABSTRACT

We consider varieties of semilattice based algebras. We define a variant of interpolation property and find its algebraic equivalent.

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NTERPOLATION AND AMALGAMTION

There is a close connection between syntactic and categorial properties in varieties of algebras. In particular, between different versions of interpolation and Beth properties in varieties of algebras and amalgamation and epimorphisms surjectivity.

One can find the definitions of amalgamation property AP, super-amalgamation SupAP, strong amalgamation StrAP, restricted amalgamation RAP, strong epimorphisms surjectivity SES and of their algebraic equivalents, namely, Robinson property ROB* (originated from H.Ono 1986), restricted interpolation IPR, projective Beth property PBP in [Gabbay, Maksimova 2005, M 2003].

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For instance, a variety V has the amalgamation property AP iff it has ROB* [Ono 1986].

(AP) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of the algebras $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $g : \mathbf{B} \rightarrow \mathbf{D}$ and $h : \mathbf{C} \rightarrow \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$.

(ROB*) Let $\Gamma(\mathbf{x}, \mathbf{y})$ and $\Delta(\mathbf{x}, \mathbf{z})$ satisfy the condition:
for all $\alpha(\mathbf{x})$, $\Gamma(\mathbf{x}, \mathbf{y}) \models_V \alpha(\mathbf{x})\}$ iff $\Delta(\mathbf{x}, \mathbf{z}) \models_V \alpha(\mathbf{x})\}$.
If $\Gamma(\mathbf{x}, \mathbf{y}), \Delta(\mathbf{x}, \mathbf{z}) \models_V \delta(\mathbf{x}, \mathbf{z})$, then $\Delta(\mathbf{x}, \mathbf{z}) \models_V \delta(\mathbf{x}, \mathbf{z})$.

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SEMILATTICE BASED VARIETIES

Let us fix an arbitrary signature consisting of functional symbols and constants and including $\wedge$. We consider semilattice based varieties of algebras, where $\wedge$ is a greatest lower bound; as usual, $x \leq y \iff x \wedge y = x$. We define a variant of interpolation property and find its algebraic equivalent.

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We recall the definition of Super-amalgamation property:
(SupAP) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of the algebras $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and, moreover,

$$g(x) \leq h(y) \iff (\exists z \in \mathbf{A})(\mathbf{x} \leq \mathbf{z} \text{ and } \mathbf{z} \leq \mathbf{y}),$$

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(WSupAP) arises from (SupAP) by changing ”monomorphisms” to ”homomorphisms”.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be pairwise disjoint lists of variables. We formulate *Inequalities interpolation principle*:
(IIP) Let $\Gamma(\mathbf{x}, \mathbf{y})$ and $\Delta(\mathbf{x}, \mathbf{z})$ satisfy the condition:
for all $\alpha(\mathbf{x})$, $\Gamma(\mathbf{x}, \mathbf{y}) \models_V \alpha(\mathbf{x})\}$ iff $\Delta(\mathbf{x}, \mathbf{z}) \models_V \alpha(\mathbf{x})\}$.
If $\Gamma(\mathbf{x}, \mathbf{y}), \Delta(\mathbf{x}, \mathbf{z}) \models_V u(\mathbf{x}, \mathbf{y}) \leq v(\mathbf{x}, \mathbf{z})$, then there is a term $t(\mathbf{x})$ such that $\Gamma(\mathbf{x}, \mathbf{y}) \models_V u(\mathbf{x}, \mathbf{y}) \leq t(\mathbf{x})$ and
$\Delta(\mathbf{x}, \mathbf{z}) \models_V t(\mathbf{x}) \leq v(\mathbf{x}, \mathbf{z})$.

Theorem

For any variety V:

1. IIP is equivalent to WSupAP;
2. V has SupAP iff it has AP and WSupAP.

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Beth's definability properties

have as their source the theorem on implicit definability proved by E.Beth in 1953 for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable. We formulate an analog of Beth's property for varieties, namely *the projective Beth property* PBP.

Let $\mathbf{x}$, $\mathbf{q}$, $\mathbf{q}'$ be disjoint lists of variables not containing y and z, $\Gamma(\mathbf{x}, \mathbf{q}, y)$ a set of equations.
(PBP) If $\Gamma(\mathbf{x}, \mathbf{q}, y), \Gamma(\mathbf{x}, \mathbf{q}', z) \models_V (y = z)$, then $\Gamma(\mathbf{x}, \mathbf{q}, y) \models_V (y = t(\mathbf{x}))$ for some term $t(\mathbf{x})$.

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M1998:

Lemma

PBP $\Longleftrightarrow$ **SES.**

(SES) For any **A**,**B** in V, for any monomorphism α : **A** $\rightarrow$ **B** and for any $x \in$ **B** $-$ α(**A**) there exist **C** $\in V$ and homomorphisms β : **B** $\rightarrow$ **C** and γ : **B** $\rightarrow$ **C** such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

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M1998:

Lemma

PBP $\iff$ **SES.**

(SES) For any **A**,**B** in V, for any monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and for any $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$ and $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

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RAP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that **A** is a common subalgebra of **B** and **C** there exist an algebra **D** in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction of δ onto **A** is a monomorphism.

Theorem

1. $IIP \Rightarrow PBP$;
2. $WSupAP \Rightarrow (SES \ \& \ RAP)$.

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RAP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction of δ onto $\mathbf{A}$ is a monomorphism.

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1. *IIP* $\Rightarrow$ *PBP*;
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VARIETIES ASSOCIATED WITH NON-CLASSICAL LOGICS

Theorem

Let a variety V satisfy the condition: there are a term $\varepsilon(x, y)$ and a constant e such that $x = y \models_V e \leq \varepsilon(x, y)$ and $e \leq \varepsilon(x, y) \models_V x = y$.
Then IIP implies AP, and so IIP, SupAP and WSupAP are equivalent.

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