Polish groups and full isometry groups

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What Polish spaces and Polish groups are

Definition

A Polish metric space (X, d) is a metric space such that the topology induced by d is Polish, that is, separable and completely metrizable.

Definition

A topological group is a group endowed with a topology that makes its group operations continuous. A Polish group is a topological group whose topology is Polish.

Examples of Polish groups

- $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$;
- The additive groups of separable Banach spaces;
- Compact or locally compact, second countable groups (e.g. Lie groups).

Various groups of transformations such as:

- for a compact metrizable space X, its full group of homeomorphisms $\mathrm{Hom}(X)$ with the uniform convergence topology;
- for a Polish metric X, its full isometry group $\mathrm{Iso}(X)$ with the pointwise convergence topology;
- Closed subgroups of the above.

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X is proper $\implies \mathrm{Iso}(X)$ is locally compact.

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Definition

A metric space (X, d) is called proper if all closed balls in it are compact.

On the other hand:

$$G \leq \mathrm{Iso}(G)$$

for any fixed left-invariant metric d on G.

Reason: By the Birkhoff-Kakutani theorem, each Polish G admits a left-invariant compatible metric, so we can identify elements in G with isometries defined by left-translations.

$$g \mapsto g(h) = gh, \; h \in G$$

This is a homeomorphic embedding.

Thus, it can be easily shown that 'weak converses' to the above facts hold:

G is Polish $\Longrightarrow G \leq \mathrm{Iso}(X)$ for some Polish space X.

G is compact $\Longrightarrow G \leq \mathrm{Iso}(X)$ for some compact X.

G is locally compact $\Longrightarrow G \leq \mathrm{Iso}(X)$ for some proper X.

Question: How about 'strong converses'?

It was known before that

G is Polish $\Longrightarrow$ $G \cong$Iso(X) for some Polish X. (Gao, Kechris)

G is compact $\Longrightarrow$ $G \cong$Iso(X) for some compact X. (Melleray)

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Theorem

Let G be a locally compact Polish group. There exists a proper Polish metric space X such that $G \cong Iso(X)$.

This answers a question posed by Gao and Kechris in [1].

Katětov functions.

A crucial element of the proof of the above theorem are properly chosen subspaces of the space $E(X)$ of all Katětov functions on X. Originally, we took a somewhat different approach that lead to a weaker version of it. In this approach we used different subspaces and new metrics on them. These metrics may be of some independent interest, so we will briefly sketch their main properties.

Definition

For a metric space (X, d), consider the space $E(X)$ of Katětov functions on X, that is, functions $f : X \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for all x, $y \in X$. The set $E(X)$ is made into a metric space with the metric defined by

$$\sup_{x \in X} |f(x) - g(x)| .$$

for $f,g \in E(X)$.

Main properties of $E(X)$:

- Space X can be viewed a subspace of $E(X)$ after identifying every $x \in X$ with its distance function $d(x, \cdot)$;
- this embedding is canonical in the sense that every isometry ϕ of X can be uniquely extended to an isometry of $E(X)$:
$$\phi^*(f) = f(\phi^{-1}(x));$$
- The mapping $\phi \to \phi^*$ is a (topological) embedding of $\mathrm{Iso}(X)$ into $\mathrm{Iso}(E(X))$.

Definition

If, for a given $f \in E(X)$, there exists $S \subseteq X$ such that

$$f(x) = \min\{f(s) + d(x, s) : s \in S\}$$

for any $x \in X$, we say that S is a support of f.

Lemma

Suppose that f is a Katětov function with compact support. Then there exists a smallest support for f, denoted by $S(f)$, that is, a set supporting f which is contained in every other support of f.

- One unpleasent (from our point of view) property of $E(X)$ is that it may be large even if X is small. For example, $E(\mathbb{R})$ is not locally compact. This is true even if we restrict our attention only to Katětov functions with finite support.

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- To deal with it, we introduce a new metric on the space of Katětov functions on X with compact support.
- Suppose that d' is another compatible metric on X. In light of the preceding lemma, we can define the following metric ρ on the set of Katětov functions on X with compact support:

$$\rho(f,g) = \sup_x \{|f(x) - g(x)|\} + d'(S(f), S(g))$$

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- This space, denoted by $E_C(X, d, d')$, is Polish.

Moreover:

Lemma

Assume that X is a locally compact space, and let d, d' be compatible metrics on X. Then $E_C(X, d, d')$ is also locally compact. Moreover, if d' is proper, so is $E_C(X, d, d')$.

Lemma

Assume that X has no isolated points. Then, for a given $\phi \in \mathrm{Iso}(X)$, the function ϕ^* is the unique extension of ϕ to an isometry of $E_C(X, d, d^{1/2})$, and the function $\phi \mapsto \phi^*$ is a continuous embedding of $\mathrm{Iso}(X)$ into $\mathrm{Iso}(E_C(X, d, d^{1/2}))$.

Ultrametric spaces and their isometry groups.

Definition

A metric space (X, d) is called ultrametric if it satisfies a strong version of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for every x,y,$z \in X$.

Every isometry group Iso(X) of an ultrametric space X is a closed subgroups of S_∞. In [1], Gao and Kechris showed that all such Iso(X) contain an element of order 2, so not all closed subgroups of S_∞ can be realized as Iso(X), X ultrametric. Another reason for this:

Theorem

S_∞ and $\mathbb{Z}_2$ are the only topologically simple non-trivial Polish groups isomorphic to the isometry group of an ultrametric space.

Definition

Let (X, d) and (Y, ρ) be metric spaces. A function $\pi : X \to Y$ is called a metric quotient function if it is surjective and for all $y, y' \in Y$

$$\rho(y, y') = \inf\{d(x, x') : x, x' \in X,\ \pi(x) = y \text{ and } \pi(x') = y'\}.$$

If π is as above, we call $y, y' \in Y$ conjugate with respect to π if there are $x, x' \in X$ such that $\pi(x) = y$, $\pi(x') = y'$ and $\rho(y, y') = d(x, x')$.

Theorem

Let (X, d) be a separable ultrametric space. There exists an ultrametric space (Ω, ρ) and a metric quotient function $\pi : X \to \Omega$ such that

(i) *π is invariant under $\mathrm{Iso}(X)$;*

(ii) *if ϕ is a partial isometry of X with finite domain and such that for each x in its domain $\pi(x) = \pi(\phi(x))$, then ϕ can be extended to an element of $\mathrm{Iso}(X)$.*

Additionally,

(iii) *if $y \in \Omega$ is conjugate with respect to π to all points in Ω, then each isometry of $\pi^{-1}(y)$ can be extended to an element of $\mathrm{Iso}(X)$.*

Two comments:

- Fibers are closed (so Polish) subsets of X, and they are ultrahomogeneous.
  Ultrahomogeneous Polish spaces are easy to classify!
- Let X be a separable ultrametric space. If there is a point whose orbit under the action of $\mathrm{Iso}(X)$ is dense, then X is ultrahomogeneous.

Question: Can isometries of the fibres be always extended to isometries of X?

Example

There exists a Polish ultrametric space with two homogeneity components and an isometry of one of the homogeneity components that does not extend to an isometry of the whole space.

I would like to thank

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