

**Automorphism Spectra and  
Tree-Definability**

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## The Automorphism Spectrum

**Definition:** For a computable structure  $\mathcal{A}$ , the *automorphism spectrum* of  $\mathcal{A}$  is the set

$$\text{AutSpec}^*(\mathcal{A}) = \{\text{deg}(f) : f \in \text{Aut}(\mathcal{A}) \ \& \ f \neq \text{id}\}.$$

So  $\text{AutSpec}^*(\mathcal{A})$  measures the complexity of the nontrivial symmetries of  $\mathcal{A}$ .

Notice that the same definition makes sense for noncomputable structures  $\mathcal{A}$  as well.

## Singleton Automorphism Spectra

The following are known automorphism spectra:

- $\{\mathbf{d}\}$ , for every c.e. degree  $\mathbf{d}$ , uniformly (HMM).
- $\{\mathbf{d}\}$ , for every degree  $\mathbf{d} \leq \mathbf{0}'$  (Hirschfeldt; Schmerl).
- $\{\mathbf{d}\}$ , for every degree  $\mathbf{d}$  with  $\mathbf{0}^{(n)} \leq \mathbf{d} \leq \mathbf{0}^{(n+1)}$  (HMM, using a technique of Marker and others).

**Theorem:** If  $\text{AutSpec}^*(\mathcal{A}) = \{\mathbf{d}\}$ , then  $\mathbf{d}$  is hyperarithmetical.

(This follows from the Perfect Set Theorem.)

## Tree-Definability

**Definition:** A function  $f : \omega \rightarrow \omega$  is *tree-definable* if there exists a computable subtree  $T \subseteq \omega^{<\omega}$  such that  $f$  is the unique (infinite) path through  $T$ .

We also say that  $\text{deg}(f)$  is *tree-definable*.

**Theorem:** The following are equivalent:

1.  $\text{AutSpec}^*(\mathcal{A})$  is at most countable.
2.  $\text{Aut}(\mathcal{A})$  is at most countable.
3. Every  $\mathbf{d} \in \text{AutSpec}^*(\mathcal{A})$  is tree-definable.

Clearly  $(1 \iff 2)$  and  $(3 \implies 1)$ .

For  $(2 \implies 3)$ , apply Kueker's Theorem to get a finite tuple  $p_1, \dots, p_n \in \mathcal{A}$  such that  $(\mathcal{A}, \vec{p})$  is rigid. So for  $f \in \text{Aut}(\mathcal{A})$ , build a computable tree of those partial automorphisms  $\varphi$  of  $\mathcal{A}$  with  $\varphi(p_i) = f(p_i)$  for all  $i$ .

## Equivalence

**Theorem:** A degree  $\mathbf{d}$  is tree-definable iff there is a computable  $\mathcal{A}$  with  $\text{AutSpec}^*(\mathcal{A}) = \{\mathbf{d}\}$ .

( $\implies$ ) follows from a construction by Morozov, 1993. (Cf. *Handbook of Recursive Mathematics*, vol. 1.)

Since all degrees  $\mathbf{0}^{(\alpha)}$  with  $\alpha < \omega_1^{CK}$  are tree-definable, this yields:

**Corollary:** For every  $\alpha < \omega_1^{CK}$ , there exists a computable  $\mathcal{A}$  with  $\text{AutSpec}^*(\mathcal{A}) = \{\mathbf{0}^{(\alpha)}\}$ .

## Outside the Bubbles

Recall that each of the following degrees

forms a singleton automorphism spectrum.

What about arithmetic degrees outside this set?

## New Automorphism Spectrum

**Theorem:** There exists a tree-definable Turing degree  $d \leq \mathbf{0}''$  which is incomparable with  $\mathbf{0}'$ .

**Corollary:** There exists a computable  $\mathcal{A}$  whose sole nontrivial symmetry is arithmetical but incomparable with  $\mathbf{0}'$ .

Proof of theorem builds a computable  $T \subseteq \omega^{<\omega}$  with unique path  $p$  satisfying:

$$\mathcal{R}_e : p \neq \Phi_e^{\emptyset'}$$

$$\mathcal{N}_e : \emptyset' \neq \Phi_e^p.$$

## Satisfying $\mathcal{R}_e$

At some node  $\rho_e$  on  $p$  we make  $p \neq \Phi_e^{\emptyset'}$ . Let  $n = |\rho_e|$  and start building  $T$  with  $p(n) = d_e$ . Each time  $\Phi_{e,s}^{\emptyset'}(n) \downarrow$  at some  $s$ , pick a new large  $y \neq \Phi_{e,s}^{\emptyset'}(n)$  and switch the construction so that  $p(n) = y$ . If later  $\emptyset'$  changes on the use of this computation, switch back to  $p(n) = d_e$ .

So exactly one node above  $\rho_e$  is active at infinitely many stages.



## Satisfying $\mathcal{N}_e$

To make  $\emptyset' \neq \Phi_e^p$ , use a Sacks preservation strategy at a node  $\nu_e$  on  $p$ :

$$l(\nu_e, s) = \max\{x : (\forall y < x) \Phi_{e,s}^p(y) \downarrow = \emptyset'_s(y)\}$$
$$r(\nu_e, s) = \max\{\text{use}(\Phi_{e,s}^p(y)) : y \leq l(\nu_e, s)\}.$$

So if  $\Phi_e^p = \emptyset'$ , then  $l(\nu_e, s) \rightarrow \infty$  and we would be able to compute  $\emptyset'$ .

For each  $\nu_e \subset p$ ,  $T$  contains only one extension of  $\nu_e$  up to the level  $\lim l(\nu_e, s)$ . (That extension will be  $\rho_{e+1}$ .) So a  $\mathbf{0}''$ -oracle can compute  $p$ , by finding the unique node at each level which is actively infinitely often.

Every time we add a new node  $\sigma$  to  $T_s$ , we choose  $\sigma(|\sigma| - 1) > s$ . So  $T = \bigcup_s T_s$  is computable.

## Outside Higher Bubbles

**Corollary:** For each  $n \in \omega$ , there exists a computable  $\mathcal{A}$  with  $\text{AutSpec}^*(\mathcal{A}) = \{\mathbf{d}\}$ , where  $\mathbf{0}^{(n)} \leq \mathbf{d} \leq \mathbf{0}^{(n+2)}$  but  $\mathbf{d}$  is incomparable with  $\mathbf{0}^{(n+1)}$ .

**Proof:** Relativize the construction above to build  $\mathcal{B} \leq \mathbf{0}^{(n)}$  with this automorphism spectrum; then apply Marker's technique to  $\mathcal{B}$ .

## Known Automorphism Spectra

- Singletons  $\{\mathbf{d}\}$ :  $\mathbf{0}^{(n)} \leq \mathbf{d} \leq \mathbf{0}^{(n+1)}$ ,  $\mathbf{d} = \mathbf{0}^{(\alpha)}$ , one  $\mathbf{d}$  outside each bubble.
- All closures under join of finite unions of existing automorphism spectra.
- One set of three pairwise-incomparable degrees  $\leq_T \mathbf{0}'$ .
- All upward closures of existing automorphism spectra.
- The closure under join of  $\{\text{deg}(W_e) : W_e \in E\}$ , for any uniformly c.e. family  $E$  of c.e. sets; likewise for properly  $\Sigma_{n+1}$  sets for other  $n$ .
- The set of all proper  $\Sigma_{n+1}$ -degrees; also the union of these for all  $n$ .