# Prime Models of Computably Enumerable Degree

Rachel Epstein

Department of Mathematics
University of Chicago

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- During the 1970's, two groups separately began to study the computability of these models:
  - Harrington, Millar, and Morley in America
  - Goncharov, Nurtazin, and Peretyat'kin in Russia

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## Definition (Prime Model)

A model $\mathcal{A}$ of a theory $T$ is *prime* if $\mathcal{A}$ elementarily embeds into every model of $T$.

For example, the algebraic numbers are a prime model of $\mathrm{ACF}_0$.

## Definition (Complete Formula)

An $n$-ary formula $\theta(\overline{x})$ consistent with $T$ is *complete* if for all $\psi(\overline{x})$, either $T \vdash (\theta \rightarrow \psi)$ or $T \vdash (\theta \rightarrow \neg\psi)$.

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A countable model $\mathcal{A}$ is atomic $\iff$ $\mathcal{A}$ is prime.

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**Definition (Decidable Theories and Models)**

- A theory $T$ is *decidable* if the set of sentences in $T$ is computable.
- A model $\mathcal{A}$ is *decidable* if its elementary diagram $D^e(\mathcal{A})$ is computable.
- A model $\mathcal{A}$ has *degree* **d** if $D^e(\mathcal{A})$ has Turing degree **d**.

Another interesting line of research is to study the atomic diagrams.

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*Every complete atomic theory has a prime model.*

The Prime Model Theorem cannot be effectivized:

**Theorem (Goncharov-Nurtazin (1974), Millar (1978))**

*There is a complete atomic decidable (CAD) theory with no decidable prime model.*

- Proof idea:
  - If $\mathcal{A}$ is a prime model, then $D^e(\mathcal{A})$ can compute a listing of the principal types of $T$.
  - Build a CAD theory $T$ so that if $\varphi_e$ appears to list the principal types, it lists some non-principal type as well.

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**Question**

*What are the degrees of prime models of CAD theories?*

**Theorem (Knight, Upward Closure, (1986))**

*For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.*

**Theorem (Denisov (1989), Drobotun (1978), Millar (1978))**

*Every CAD theory has a prime model computable in $\mathbf{0}'$.*

- $\mathbf{0}'$ can carry out the usual construction of a prime model.
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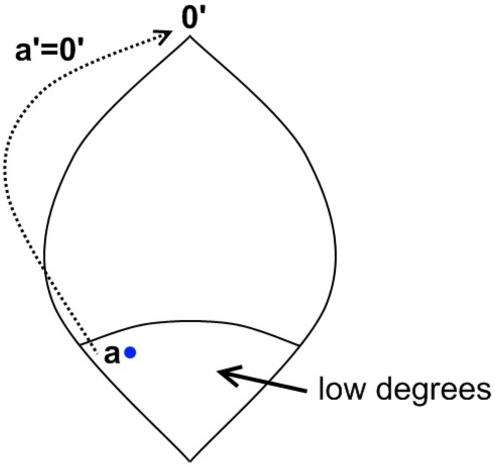
## Theorem (Csima, Prime Low Basis Theorem (2004))

*Every CAD theory has a prime model $\mathcal{A}$ of low degree, i.e., $D^e(\mathcal{A})' \equiv_T \mathbf{0}'$.*



# Computably Enumerable Sets

**Definition**

A set $C$ is *computably enumerable* (c.e.) if there is an effective listing of its elements.

- Computably enumerable sets and degrees have played a central role in computability theory since Post's Problem of 1944.
- There are many applications of the c.e. sets, including:
  - Unsolvability of Hilbert's 10th problem
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**Remark**

*If $D^e(\mathcal{A})$ is a c.e.* **set**, *then it is computable.*

We will instead focus our attention on prime models of computably enumerable Turing **degree**.

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A Turing degree **c** is *computably enumerable* (c.e.) if **c** contains a c.e. set.

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**Theorem (Csima, Hirschfeldt, Knight, Soare, Prime Bounding (2004))**

*The degree $\mathbf{d} \leq \mathbf{0}'$ is the degree of a prime model of every CAD theory $\iff \mathbf{d}'' > \mathbf{0}''$ (i.e, $\mathbf{d}$ is nonlow$_2$).*

**Corollary**

*All nonlow$_2$ c.e. degrees are the degrees of prime models for every CAD theory.*

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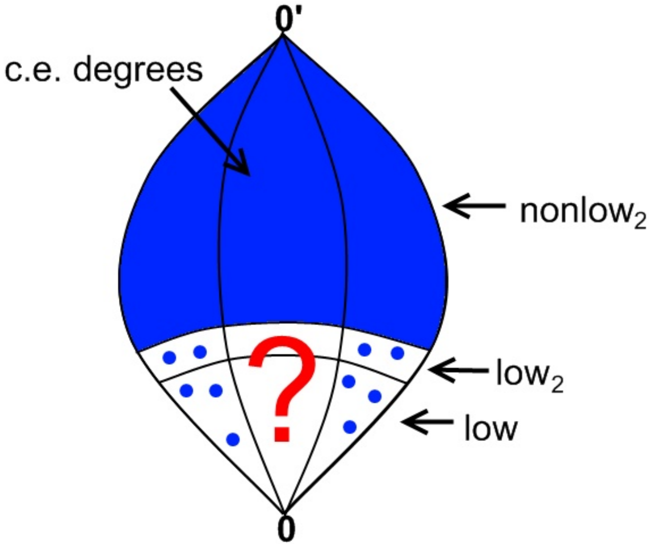


blue = degrees of prime models for a given CAD theory

# Prime Low C.E. Basis Theorem

**Theorem (Epstein)**

*Every CAD theory has a prime model of low c.e. degree.*

- The proof differs greatly from Csima's proof for low degrees:
  - Csima used a forcing argument with a $\mathbf{0}'$-oracle construction.
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# Density of Prime Models of C.E. Degree

**Theorem (Sacks, Density Theorem)**

*Let* $\mathbf{d}$ *and* $\mathbf{c}$ *be c.e. degrees with* $\mathbf{d} < \mathbf{c}$. *Then there is a c.e. degree* $\mathbf{b}$ *between* $\mathbf{d}$ *and* $\mathbf{c}$.

**Theorem (Epstein)**

*Let* $T$ *be a CAD theory,* $\mathbf{c}$ *be nonlow*$_2$ *and c.e., and* $\mathbf{d} < \mathbf{c}$ *be c.e. Then there is a prime model* $\mathcal{A}$ *of* $T$ *of c.e. degree* $\mathbf{a}$ *with* $\mathbf{d} < \mathbf{a} < \mathbf{c}$ *and* $\mathbf{a}' = \mathbf{d}'$.

- Uses infinite injury
- Below any nonlow$_2$ c.e. degree, there is a prime model of low c.e. degree.
- Can make $\mathbf{a}'$ any degree c.e. in $\mathbf{c}$ and above $\mathbf{d}'$.

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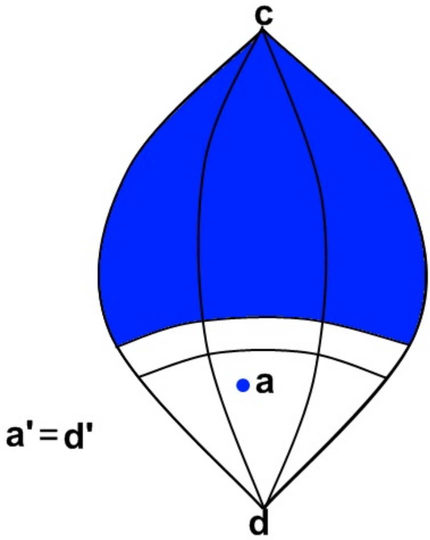
c
a
a'=d'
d

# Pushing Down the Degree of a Prime Model

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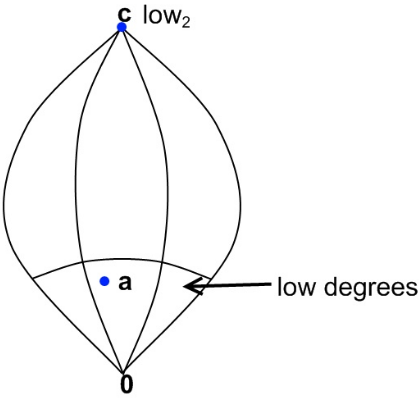
*Let* $\mathbf{c} > \mathbf{0}$ *be the c.e. degree of a prime model of a CAD theory* $T$*. Then there is a prime model of* $T$ *of low c.e. degree* $\mathbf{a} < \mathbf{c}$.

This theorem does not hold for non-c.e. degrees.



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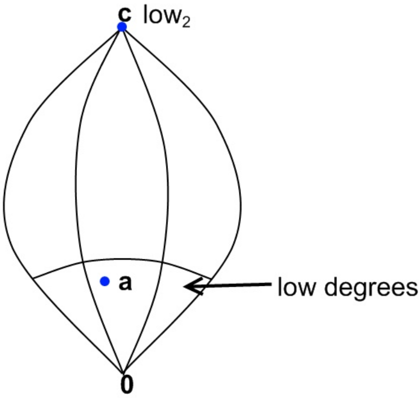
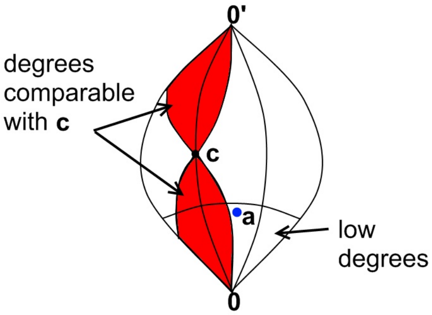


## Corollary

*For any degree* $\mathbf{c}$*, with* $\mathbf{0} < \mathbf{c} < \mathbf{0}'$*, every CAD theory* $T$ *has a prime model of low c.e. degree* $\mathbf{a}$ *such that* $\mathbf{c}|\mathbf{a}$ *(i.e.,* $\mathbf{c} \not\leq \mathbf{a}$ *and* $\mathbf{a} \not\leq \mathbf{c}$*).*

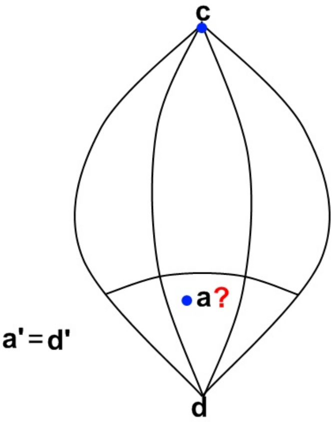


**Corollary**

*Every CAD $T$ has a minimal pair of prime models of low c.e. degree.*

## Question

*Can we combine the "pushdown" and "density" theorems? That is, given a prime model of c.e. (low$_2$) degree* $\mathbf{c}$*, and given a c.e. degree* $\mathbf{d} < \mathbf{c}$*, is there a prime model of c.e. degree* $\mathbf{a}$ *between* $\mathbf{d}$ *and* $\mathbf{c}$*? Can we guarantee that* $\mathbf{a}' = \mathbf{d}'$*?*



# For Further Reading

B. F. Csima.
Degree Spectra of Prime Models.
*Journal of Symbolic Logic*, 69:430–442, 2004.

B. F. Csima, D. R. Hirschfeldt, J. F. Knight, and R. I. Soare.
Bounding Prime Models.
*Journal of Symbolic Logic*, 69:1117–1142, 2004.

R. Epstein
Prime Models of Computably Enumerable Degree.
*Journal of Symbolic Logic*, to appear.

Thank you for listening.