Prime Models of Computably Enumerable Degree

Rachel Epstein

Department of Mathematics
University of Chicago

July 3, 2008 / Logic Colloquium, Bern
In 1961, Robert Vaught introduced *prime, homogeneous,* and *saturated* models.

During the 1970’s, two groups separately began to study the computability of these models:
- Harrington, Millar, and Morley in America
- Goncharov, Nurtazin, and Peretyat’kin in Russia

For this talk, all languages, theories, and structures are countable, and all theories are complete.
In 1961, Robert Vaught introduced *prime*, *homogeneous*, and *saturated* models.

During the 1970’s, two groups separately began to study the computability of these models:
- Harrington, Millar, and Morley in America
- Goncharov, Nurtazin, and Peretyat’kin in Russia

For this talk, all languages, theories, and structures are countable, and all theories are complete.
In 1961, Robert Vaught introduced *prime*, *homogeneous*, and *saturated* models.

During the 1970’s, two groups separately began to study the computability of these models:

- Harrington, Millar, and Morley in America
- Goncharov, Nurtazin, and Peretyat’kin in Russia

For this talk, all languages, theories, and structures are countable, and all theories are complete.
**Definition (Prime Model)**

A model $\mathcal{A}$ of a theory $T$ is *prime* if $\mathcal{A}$ elementarily embeds into every model of $T$.

For example, the algebraic numbers are a prime model of $\text{ACF}_0$.

**Definition (Complete Formula)**

An $n$-ary formula $\theta(\overline{x})$ consistent with $T$ is *complete* if for all $\psi(\overline{x})$, either $T \vdash (\theta \rightarrow \psi)$ or $T \vdash (\theta \rightarrow \neg \psi)$.

**Definition (Atomic Model)**

A model $\mathcal{A}$ of $T$ is *atomic* if each tuple in $\mathcal{A}$ satisfies a complete formula. Thus, an atomic model realizes only the principal types of $T$.

A countable model $\mathcal{A}$ is atomic $\iff \mathcal{A}$ is prime.
Definition (Prime Model)
A model $\mathcal{A}$ of a theory $T$ is *prime* if $\mathcal{A}$ elementarily embeds into every model of $T$.

For example, the algebraic numbers are a prime model of $ACF_0$.

Definition (Complete Formula)
An $n$-ary formula $\theta(x)$ consistent with $T$ is *complete* if for all $\psi(x)$, either $T \vdash (\theta \rightarrow \psi)$ or $T \vdash (\theta \rightarrow \neg \psi)$.

Definition (Atomic Model)
A model $\mathcal{A}$ of $T$ is *atomic* if each tuple in $\mathcal{A}$ satisfies a complete formula. Thus, an atomic model realizes only the principal types of $T$.

A countable model $\mathcal{A}$ is atomic $\iff \mathcal{A}$ is prime.
Definition (Prime Model)
A model $\mathcal{A}$ of a theory $T$ is *prime* if $\mathcal{A}$ elementarily embeds into every model of $T$.

For example, the algebraic numbers are a prime model of $\text{ACF}_0$.

Definition (Complete Formula)
An $n$-ary formula $\theta(x)$ consistent with $T$ is *complete* if for all $\psi(x)$, either $T \vdash (\theta \rightarrow \psi)$ or $T \vdash (\theta \rightarrow \neg \psi)$.

Definition (Atomic Model)
A model $\mathcal{A}$ of $T$ is *atomic* if each tuple in $\mathcal{A}$ satisfies a complete formula. Thus, an atomic model realizes only the principal types of $T$.

A countable model $\mathcal{A}$ is atomic $\iff \mathcal{A}$ is prime.
**Definition (Prime Model)**

A model $\mathcal{A}$ of a theory $T$ is *prime* if $\mathcal{A}$ elementarily embeds into every model of $T$.

For example, the algebraic numbers are a prime model of $\mathbb{ACF}_0$.

**Definition (Complete Formula)**

An $n$-ary formula $\theta(\bar{x})$ consistent with $T$ is *complete* if for all $\psi(\bar{x})$, either $T \vdash (\theta \rightarrow \psi)$ or $T \vdash (\theta \rightarrow \neg \psi)$.

**Definition (Atomic Model)**

A model $\mathcal{A}$ of $T$ is *atomic* if each tuple in $\mathcal{A}$ satisfies a complete formula. Thus, an atomic model realizes only the principal types of $T$.

A countable model $\mathcal{A}$ is atomic $\iff \mathcal{A}$ is prime.
Definition (Atomic Theory)

A theory $T$ is atomic if every formula consistent with $T$ can be extended to a complete formula.

Theorem (Prime Model Theorem, Vaught)

Every complete atomic theory has a prime model.

How can we effectivize this theorem?
Definition (Atomic Theory)
A theory $T$ is atomic if every formula consistent with $T$ can be extended to a complete formula.

Theorem (Prime Model Theorem, Vaught)
Every complete atomic theory has a prime model.

How can we effectivize this theorem?
Definition (Atomic Theory)
A theory $T$ is atomic if every formula consistent with $T$ can be extended to a complete formula.

Theorem (Prime Model Theorem, Vaught)
Every complete atomic theory has a prime model.

How can we effectivize this theorem?
**Definition (Decidable Theories and Models)**

- A theory \( T \) is *decidable* if the set of sentences in \( T \) is computable.
- A model \( \mathcal{A} \) is *decidable* if its elementary diagram \( D^e(\mathcal{A}) \) is computable.
- A model \( \mathcal{A} \) has degree \( d \) if \( D^e(\mathcal{A}) \) has Turing degree \( d \).

Another interesting line of research is to study the atomic diagrams.
Theorem (Prime Model Theorem, Vaught)

Every complete atomic theory has a prime model.

The Prime Model Theorem cannot be effectivized:

Theorem (Goncharov-Nurtazin (1974), Millar (1978))

There is a complete atomic decidable (CAD) theory with no decidable prime model.

Proof idea:

- If $\mathcal{A}$ is a prime model, then $D^e(\mathcal{A})$ can compute a listing of the principal types of $T$.
- Build a CAD theory $T$ so that if $\varphi_e$ appears to list the principal types, it lists some non-principal type as well.
Theorem (Prime Model Theorem, Vaught)

*Every complete atomic theory has a prime model.*

The Prime Model Theorem cannot be effectivized:

Theorem (Goncharov-Nurtazin (1974), Millar (1978))

*There is a complete atomic decidable (CAD) theory with no decidable prime model.*

**Proof idea:**

- If $\mathcal{A}$ is a prime model, then $D^e(\mathcal{A})$ can compute a listing of the principal types of $T$.
- Build a CAD theory $T$ so that if $\varphi_e$ appears to list the principal types, it lists some non-principal type as well.
Theorem (Prime Model Theorem, Vaught)

Every complete atomic theory has a prime model.

The Prime Model Theorem cannot be effectivized:

Theorem (Goncharov-Nurtazin (1974), Millar (1978))

There is a complete atomic decidable (CAD) theory with no decidable prime model.

Proof idea:

- If $\mathcal{A}$ is a prime model, then $D^e(\mathcal{A})$ can compute a listing of the principal types of $T$.
- Build a CAD theory $T$ so that if $\varphi_e$ appears to list the principal types, it lists some non-principal type as well.
Question

What are the degrees of prime models of CAD theories?

Theorem (Knight, Upward Closure, (1986))

For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.

Theorem (Denisov (1989), Drobotun (1978), Millar (1978))

Every CAD theory has a prime model computable in $0'$. 

- $0'$ can carry out the usual construction of a prime model.
- Many people have worked on improving this.
What are the degrees of prime models of CAD theories?

For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.

Every CAD theory has a prime model computable in $0'$. $0'$ can carry out the usual construction of a prime model. Many people have worked on improving this.
Question

What are the degrees of prime models of CAD theories?

Theorem (Knight, Upward Closure, (1986))

For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.

Theorem (Denisov (1989), Drobotun (1978), Millar (1978))

Every CAD theory has a prime model computable in $0'$.  

- $0'$ can carry out the usual construction of a prime model.
- Many people have worked on improving this.
Question

What are the degrees of prime models of CAD theories?

Theorem (Knight, Upward Closure, (1986))

For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.

Theorem (Denisov (1989), Drobotun (1978), Millar (1978))

Every CAD theory has a prime model computable in $0'$.  

- $0'$ can carry out the usual construction of a prime model.
- Many people have worked on improving this.
Question

What are the degrees of prime models of CAD theories?

Theorem (Knight, Upward Closure, (1986))

For a nontrivial theory $T$, the set of degrees of prime models of $T$ is upward closed.

Theorem (Denisov (1989), Drobotun (1978), Millar (1978))

Every CAD theory has a prime model computable in $0'$.  

- $0'$ can carry out the usual construction of a prime model.
- Many people have worked on improving this.
Theorem (Csima, Prime Low Basis Theorem (2004))

Every CAD theory has a prime model $\mathcal{A}$ of low degree, i.e., $D^e(\mathcal{A})' \equiv_T 0'$. 
A set $C$ is *computably enumerable* (c.e.) if there is an effective listing of its elements.

- Computably enumerable sets and degrees have played a central role in computability theory since Post’s Problem of 1944.
- There are many applications of the c.e. sets, including:
  - Unsolvability of Hilbert’s 10th problem
  - Unsolvability of word problem for groups
  - Differential geometry
A set $C$ is *computably enumerable* (c.e.) if there is an effective listing of its elements.

- Computably enumerable sets and degrees have played a central role in computability theory since Post’s Problem of 1944.
- There are many applications of the c.e. sets, including:
  - Unsolvability of Hilbert’s 10th problem
  - Unsolvability of word problem for groups
  - Differential geometry
A set $C$ is *computably enumerable* (c.e.) if there is an effective listing of its elements.

- Computably enumerable sets and degrees have played a central role in computability theory since Post's Problem of 1944.
- There are many applications of the c.e. sets, including:
  - Unsolvability of Hilbert’s 10th problem
  - Unsolvability of word problem for groups
  - Differential geometry
Remark

If $D_e(A)$ is a c.e. set, then it is computable.

We will instead focus our attention on prime models of computably enumerable Turing degree.

Definition

A Turing degree $c$ is *computably enumerable* (c.e.) if $c$ contains a c.e. set.
Remark

If $D^e(A)$ is a c.e. set, then it is computable.

We will instead focus our attention on prime models of computably enumerable Turing degree.

Definition

A Turing degree $c$ is *computably enumerable* (c.e.) if $c$ contains a c.e. set.
Theorem (Csima, Hirschfeldt, Knight, Soare, Prime Bounding (2004))

The degree $d \leq 0'$ is the degree of a prime model of every CAD theory $\iff d'' > 0''$ (i.e., $d$ is nonlow$_2$).

Corollary

All nonlow$_2$ c.e. degrees are the degrees of prime models for every CAD theory.
Theorem (Csima, Hirschfeldt, Knight, Soare, Prime Bounding (2004))

The degree $d \leq 0'$ is the degree of a prime model of every CAD theory $\iff d'' > 0''$ (i.e, $d$ is nonlow$_2$).

Corollary

All nonlow$_2$ c.e. degrees are the degrees of prime models for every CAD theory.
blue = degrees of prime models for a given CAD theory
Theorem (Epstein)

*Every CAD theory has a prime model of low c.e. degree.*

- The proof differs greatly from Csima’s proof for low degrees:
  - Csima used a forcing argument with a $0'$-oracle construction.
  - For the c.e. case, we use a priority argument with a computable construction.
Prime Low C.E. Basis Theorem

Theorem (Epstein)

*Every CAD theory has a prime model of low c.e. degree.*

- The proof differs greatly from Csima’s proof for low degrees:
  - Csima used a forcing argument with a $0'$-oracle construction.
  - For the c.e. case, we use a priority argument with a computable construction.
Density of Prime Models of C.E. Degree

Theorem (Sacks, Density Theorem)

Let $d$ and $c$ be c.e. degrees with $d < c$. Then there is a c.e. degree $b$ between $d$ and $c$.

Theorem (Epstein)

Let $T$ be a CAD theory, $c$ be nonlow$_2$ and c.e., and $d < c$ be c.e. Then there is a prime model $A$ of $T$ of c.e. degree $a$ with $d < a < c$ and $a' = d'$.

- Uses infinite injury
- Below any nonlow$_2$ c.e. degree, there is a prime model of low c.e. degree.
- Can make $a'$ any degree c.e. in $c$ and above $d'$. 

Rachel Epstein
Prime Models of Computably Enumerable Degree
Theorem (Sacks, Density Theorem)

Let \( d \) and \( c \) be c.e. degrees with \( d < c \). Then there is a c.e. degree \( b \) between \( d \) and \( c \).

Theorem (Epstein)

Let \( T \) be a CAD theory, \( c \) be nonlow\(_2\) and c.e., and \( d < c \) be c.e. Then there is a prime model \( A \) of \( T \) of c.e. degree \( a \) with \( d < a < c \) and \( a' = d' \).

- Uses infinite injury
- Below any nonlow\(_2\) c.e. degree, there is a prime model of low c.e. degree.
- Can make \( a' \) any degree c.e. in \( c \) and above \( d' \).
a' = d'
Theorem (Epstein)

Let $c > 0$ be the c.e. degree of a prime model of a CAD theory $T$. Then there is a prime model of $T$ of low c.e. degree $a < c$.

This theorem does not hold for non-c.e. degrees.
Theorem (Epstein)

Let $c > 0$ be the c.e. degree of a prime model of a CAD theory $T$. Then there is a prime model of $T$ of low c.e. degree $a < c$.

This theorem does not hold for non-c.e. degrees.
Corollary

For any degree $c$, with $0 < c < 0'$, every CAD theory $T$ has a prime model of low c.e. degree $a$ such that $c|a$ (i.e., $c \not< a$ and $a \not< c$).
Corollary

Every CAD $T$ has a minimal pair of prime models of low c.e. degree.
Question

Can we combine the “pushdown” and “density” theorems? That is, given a prime model of c.e. (low\(_2\)) degree \(c\), and given a c.e. degree \(d < c\), is there a prime model of c.e. degree \(a\) between \(d\) and \(c\)? Can we guarantee that \(a' = d'\)?
For Further Reading


Thank you for listening.