Lifschitz Realizability for Set Theories

Michael Rathjen

School of Mathematics
University of Leeds

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Independence and equiconsistency results in classical set theory
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1. The Constructible Hierarchy, $L$ (Gödel)
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2. **Forcing** (Cohen)
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2. Forcing (Cohen)
   - $\mathbb{P}$ partial order, $M$ model of set theory, $\mathbb{P} \in M$, $G$ filter on $\mathbb{P}$ and generic over $M$. $M[G]$ generic extension of $M$.
   - Permutation models for proving the independence of AC. Alternatively take $\text{HOD}^{M[G]}$ with suitably chosen $\mathbb{P}$ (homogeneous).
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5. Kripke models
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4. **Topological** models
5. **Kripke** models
6. **Categorical** models, **Topoi**, Algebraic Set Theory
7. **Proof-theoretic** methods
Two types of set existence axioms

- Explicit set existence axioms: e.g. Separation, Replacement, Exponentiation
- Non-explicit set existence axioms: e.g. Axioms of Choice
- Non-explicit set existence axioms in intuitionistic set theory: e.g. Axioms of Choice, (Strong) Collection, Subset Collection

Theorem: (Lubarsky, R)
There is a model of CZF $\text{Exp}$ in which the Dedekind reals form a proper class. So Subset Collection is necessary to show that $R^d$ is a set.
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Kleene realizability

Write $e \bullet n$ for $\{e\}(n)$. $\langle , \rangle$ is a primitive recursive and bijective pairing function on $\mathbb{N}$. Let $(e)_0 = n$ and $(e)_1 = k$ where $n, k$ are uniquely determined by $e = \langle n, k \rangle$.
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- $e \vdash^K A$ iff $A$ is true for atomic $A$.
- $e \vdash^K A \land B$ iff $(e)_0 \vdash^K A$ and $(e)_1 \vdash^K B$
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Write $e \cdot n$ for $\{e\}(n)$. $\langle , \rangle$ is a primitive recursive and bijective pairing function on $\mathbb{N}$. Let $(e)_0 = n$ and $(e)_1 = k$ where $n, k$ are uniquely determined by $e = \langle n, k \rangle$.

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- $e \vdash_K \exists x F(x)$ iff $(e)_1 \vdash_K F((e)_0)$. 
Let \((M, \cdot)\) be a structure equipped with a partial operation, that is, \(\cdot\) is a binary function with domain a subset of \(M \times M\) and co-domain \(M\).

We often omit the sign “\(\cdot\)" and adopt the convention of “association to the left". Thus \(exy\) means \((e \cdot x) \cdot y\). We also sometimes write \(e \cdot x\) in functional notation as \(e(x)\).
Definition:

A PCA is a structure \((M, \cdot)\), where \(\cdot\) is a partial binary operation on \(M\), such that \(M\) has at least two elements and there are elements \(k\) and \(s\) in \(M\) such that \(kxy\) and \(sxy\) are always defined, and

\((i)\) \(kxy = x\)

\((ii)\) \(sxyz \simeq xz(yz)\),

where \(\simeq\) means that the left hand side is defined iff the right hand side is defined, and if one side is defined then both sides yield the same result.

\((M, \cdot)\) is a total PCA if \(a \cdot b\) is defined for all \(a, b \in M\).
**Definition:**

Partial combinatory algebras are best described as the models of a formal theory **PCA**. The language of **PCA** has two distinguished constants \( k \) and \( s \). To accommodate the partial operation in a standard first order language, the language of **PCA** has a trinary relation symbol \( \text{Ap} \). The terms of **PCA** are just the variables and constants. \( \text{Ap} \) will almost never appear in what follows as we prefer to write \( t_1 t_2 \simeq t_3 \) for \( \text{Ap}(t_1, t_2, t_3) \). In order to facilitate the formulation of the axioms, the language of **PCA** is expanded definitionally with the symbol \( \simeq \) and the auxiliary notion of an application term or partial term is introduced.
In this paper, the logic of PCA is assumed to be that of intuitionistic predicate logic with identity. PCA’s non-logical axioms are the following:

**Axioms of PCA**

1. \( ab \simeq c_1 \land ab \simeq c_2 \rightarrow c_1 = c_2 \).
2. \((kab) \downarrow \land kab \simeq a\).
3. \((sab) \downarrow \land sabc \simeq ac(bc)\).
4. \(k \neq s\).
The following shows how $\lambda$-terms can be constructed in $\text{PCA}$.

**Lemma:**

For each application term $t$ and variable $x$, one can construct a term $\lambda x.t$, whose free variables are those of $t$, excluding $x$, such that $\text{PCA} \vdash \lambda x.t \downarrow$ and $\text{PCA} \vdash (\lambda x.t)u \simeq t[x/u]$ for all application terms $u$, where $t[x/u]$ results from $t$ by replacing $x$ in $t$ throughout by $u$.

**Proof:** We proceed by induction on the buildup of $t$. (i) $\lambda x.x$ is $\text{skk}$; (ii) $\lambda x.t$ is $\text{k}t$ for $t$ a constant of $\text{PCA}$ or variable other than $x$; (iii) $\lambda x.t_1 t_2$ is $s(\lambda x.t_1)(\lambda x.t_2)$.

$\Box$
Having $\lambda$-terms on hand, one can easily prove the recursion or fixed point theorem in $\text{PCA}$, and consequently that all recursive functions are definable in $\text{PCA}$. The elegance of the combinator arises from the fact that, at least in theory, anything that can be done in a programming language can be done using solely $k$ and $s$.

**Lemma:**

*Recursion Theorem* There is an application term $r$ such that $\text{PCA}$ proves:

$$rx \downarrow \land rxy \simeq x(rx)y.$$  

**Proof:** Take $r$ to be $\lambda x.gg$ where $g := \lambda y.x(yy)y$. Then $rx \simeq gg \simeq (\lambda y.x(yy)y)g \simeq \lambda y.x(gg)y$, so that $rx \downarrow$ by Lemma 0.4. Moreover, $rxy \simeq x(gg)y \simeq x(rx)y$.  

**Corollary:**

$$\text{PCA} \vdash \forall f \exists g \forall x_1 \ldots \forall x_n g(x_1, \ldots, x_n) \simeq f(g, x_1, \ldots, x_n).$$
Recursion-theoretic Examples of Combinatory Algebras

The primordial PCA is furnished by Turing machine application on the integers. There are many other interesting PCAs that provide us with a laboratory for the study of computability theory. As the various definitions are lifted to more general domains and notions of application other than Turing machine applications some of the familiar results break down. By studying the notions in the general setting one sees with a clearer eye the truths behind the results on the integers.
Kleene’s first model
The “standard” applicative structure is Kleene’s first model, called $K_1$, in which the universe $|K_1|$ is $\mathbb{N}$ and $Ap^{K_1}(x, y, z)$ is Turing machine application:

$$Ap^{K_1}(x, y, z) \iff \{x\}(y) \simeq z.$$  

The primitive constants of $PCA^+$ are interpreted over $\mathbb{N}$ in the obvious way, and $N$ is interpreted as $\mathbb{N}$.

Relative Recursion
Let $f$ be any partial function from $\mathbb{N}$ to $\mathbb{N}$. Consider the recursive functions relative to $f$. They are defined by means of Turing machines with oracles; if the oracle asks for a value of $f$ that is not defined, the whole computation diverges. Define $e \cdot x := \{e\}^f(x)$. 

LIFSCHITZ REALIZABILITY FOR SET THEORIES
Nonstandard Models of Peano Arithmetic

Let $M$ be a nonstandard model of Peano Arithmetic. Then $M$ can be rendered a model of $\text{APP}$ in two ways. The first is to interpret $N$ as all of $M$, i.e. $|M|$. Application is defined by,

$$e \cdot a \equiv b \text{ if } M \models \exists x [T(e, a, x) \wedge U(x) = b],$$

where $T$ and $U$ stand for Kleene’s $T$-predicate and result extracting function, respectively.

The second way is to interpret $N$ as the standard integers of $M$ and define application as before. In this model, the integers don’t form a decidable set, i.e. there is no element $e$ of $|M|$ such that $ex = 0$ or $ex = 1$ for all $x$ in $|M|$ and $ex = 0$ iff $x$ is a standard integer of $M$. 

Recursion in type-2 functionals

There is a notion of recursion relative to a given type-2 functional $I : \mathbb{N} \mathbb{N} \to \mathbb{N}$. The inductive definition of partial functions with domain a subset of $\mathbb{N}^k$ and values in $\mathbb{N}$ that are partial recursive in $I$ comprises the clauses for the ordinary partial recursive functions plus the following: If $f$ is a partial function from $\mathbb{N}^{k+1}$ to $\mathbb{N}$ which is partial recursive in $I$, then the partial function $g : \mathbb{N}^k \to \mathbb{N}$ defined by

$$g(n_1, \ldots, n_k) \simeq m \quad \text{iff} \quad \forall n \ f(n_1, \ldots, n_k, n) \downarrow \land m = I(\Lambda n.f(n_1, \ldots, n_k, n))$$

is partial recursive in $I$, where $\Lambda n.f(n_1, \ldots, n_k, n)$ denotes the function which maps $n$ to $f(n_1, \ldots, n_k, n)$.

Using integer indices for these functions, one arrives at a model of $\text{APP}$ with application $e \cdot x$ defined by $\{e\}^I(x)$. 
Kleene’s second model

The universe of “Kleene’s second model” of $\text{APP}$, $K_2$, is $\mathbb{N}\mathbb{N}$. The most interesting feature of $K_2$ is that in the type structure over $K_2$, every type-2 functional is continuous. We shall use $\alpha, \beta, \gamma, \ldots$ as variables ranging over functions from $\mathbb{N}$ to $\mathbb{N}$. In order to describe this PCA, it will be necessary to review some terminology.

**Definition:**

We assume that every integer codes a finite sequence of integers. For finite sequences $\sigma$ and $\tau$, $\sigma \subset \tau$ means that $\sigma$ is an initial segment of $\tau$; $\sigma \ast \tau$ is concatenation of sequences; $\langle \rangle$ is the empty sequence; $\langle n_0, \ldots, n_k \rangle$ displays the elements of a sequence; if this sequence is $\tau$ then $lh(\tau) = k + 1$ (read “length of $\tau$”); $\bar{\alpha}(m) = \langle \alpha(0), \ldots, \alpha(m - 1) \rangle$ if $m > 0$; $\bar{\alpha}(0) = \langle \rangle$. A function $\alpha$ and an integer $n$ produce a new function $\langle n \rangle \ast \alpha$ which is the function $\beta$ with $\beta(0) = n$ and $\beta(k + 1) = \alpha(k)$. 
Application requires the following operations on $\mathbb{N}_\mathbb{N}$:

\[
\alpha(\beta) \simeq \alpha(\mu n.\alpha(\bar{\beta}(n) > 0)) - 1
\]

i.e., $\alpha(\beta) = m$ iff $\exists n [\alpha(\bar{\beta}n) = m + 1 \land \forall i < n \alpha(\bar{\beta}i) = 0]$

\[
(\alpha|\beta)(n) = \alpha(\langle n \rangle \ast \beta)
\]

We would like to define application on $\mathbb{N}_\mathbb{N}$ by $\alpha|\beta$, but this is in general only a partial function, therefore we set:

\[
\alpha \cdot \beta = \gamma \quad \text{iff} \quad \forall n (\alpha|\beta)(n) = \gamma(n).
\]  

(1)

**Theorem:**

$K_2$ is a model of APP.
The general realizability structure

\( \mathcal{A} \) will be assumed to be a fixed but arbitrary PCA whose domain is denoted by \( |\mathcal{A}| \). \( \mathcal{P}(X) \) stands for the power set of \( X \). Ordinals are transitive sets whose elements are transitive also. We use lower case Greek letters to range over ordinals. For \( \mathcal{A} \models \text{APP} \),

\[
V(\mathcal{A})_\alpha = \bigcup_{\beta \in \alpha} \mathcal{P}(|\mathcal{A}| \times V(\mathcal{A})_\beta). \tag{2}
\]

\[
V(\mathcal{A}) = \bigcup_{\alpha} V(\mathcal{A})_\alpha. \tag{3}
\]

If \( a \in V(\mathcal{A}) \) and \( x \in a \), then \( x \)

\[
x = \langle e, b \rangle
\]

for some \( e \in |\mathcal{A}| \) and \( b \in V(\mathcal{A}) \).
This type of realizability goes back to Kreisel and Troelstra (for second order arithmetic). It was extended to set theory by Friedman and Beeson. The final step to extensional set theory was taken by McCarty. This concerns the atomic case and is basically the same as for boolean valued models (forcing).
Definition:

Let \( a, b \in V(\mathcal{A}) \) and \( e \in |\mathcal{A}| \).

- \( e \models \phi \land \psi \iff (e)_0 \models \phi \land (e)_1 \models \psi \)
- \( e \models \phi \lor \psi \iff [(e)_0 = 0 \land (e)_1 \models \phi] \lor [(e)_0 = 1 \land (e)_1 \models \psi] \)
- \( e \models \neg \phi \iff \forall f \in |\mathcal{A}| \neg f \models \phi \)
- \( e \models \phi \rightarrow \psi \iff \forall f \in |\mathcal{A}| [f \models \phi \rightarrow ef \models \psi] \)
- \( e \models \forall x \phi \iff \forall c \in V(\mathcal{A}) \ e \models \phi[x/c] \)
- \( e \models \exists x \phi \iff \exists c \in V(\mathcal{A}) \ e \models \phi[x/c] \)
The atomic cases

Definition:

\[ e \models a \in b \iff \exists c \left[ (e)_0 c \in b \land (e)_1 \models a = c \right] \]

\[ e \models a = b \iff \forall f, d \left[ (f, d) \in a \rightarrow (e)_0 f \models d \in b \right] \]
\[ \land \left[ (f, d) \in b \rightarrow (e)_1 f \models d \in a \right] \]
The atomic cases

**Definition:**

\[ e \models a \in b \text{ iff } \exists c \left[ \langle (e)_0, c \rangle \in b \land (e)_1 \models a = c \right] \]

\[ e \models a = b \text{ iff } \forall f, d \left[ (\langle f, d \rangle \in a \rightarrow (e)_0 f \models d \in b) \land (\langle f, d \rangle \in b \rightarrow (e)_1 f \models d \in a) \right] \]

**Corollary:**

**Bounded Quantifiers**

\[ e \models \forall x \in a \phi \text{ iff } \forall \langle f, c \rangle \in a ef \models \phi[x/c] \]

\[ e \models \exists x \in a \phi \text{ iff } \exists c \left[ \langle (e)_0, c \rangle \in a \land (e)_1 \models \phi[x/c] \right] \]
Theorem: (Friedman, McCarty, Beeson)

(i) \( V(\mathcal{A}) \models \text{IZF} + \text{AC}_{\omega,\omega} \)

(ii) \( V(K_1) \models \text{ECT}_0 + \text{MP} + \text{Uniformity Principle} + \text{Unzerlegbarkeit} \)

(iii) If \( \text{AC}_{\omega} \) or \( \text{DC} \) or the presentation axioms holds in the background universe then it holds in \( V(K_1) \).
• $V(K_1) \forces \text{Russian Constructivism}$
• $V(K_1) \models$ Russian Constructivism

• $V(K_2) \models$ Brouwer’s Intuitionism
Theorem: (R)

(i) Generic realizability is a self-validating semantics for CZF and CZF + REA, i.e.

\[ \text{CZF} \vdash \varphi \Rightarrow \text{CZF} \vdash V(A) \models \varphi + \text{AC}_{\omega,\omega}. \]

(ii) (CZF)

\[ V(K_1) \models \text{ECT}_0 + \text{Uniformity Principle} + \text{Unzerlegbarkeit} \]

(ii) CZF + AC_\omega \vdash "V(K_1) \models AC_\omega"

CZF + DC_\omega \vdash "V(K_1) \models DC_\omega" CZF + Presentation Axiom \vdash "V(K_1) \models Presentation Axiom".

(iii) CZF can be replaced by CZF + REA in the above.
Dragalin’s question 1970

Is what is known as Church’s thesis in intuitionistic mathematics, i.e. the schema

\[(CT_0) \forall n \exists k A(n, k) \rightarrow \exists e \forall n \{e\}(n) \downarrow \land A(n, \{e\}(n))\]

stronger than \[(CT_0!) \forall n \exists! k A(n, k) \rightarrow \exists e \forall n \{e\}(n) \downarrow \land A(n, \{e\}(n))\]

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$\text{AC}_{\omega,\omega} \text{ aka AC-NN and AC}_0$
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and $\text{CT}_0!$. 
• There is a closed instance of CT₀ which HA + CT₀ does not prove.

• More precisely, this instance can be taken to be of the form

$$\forall n \exists k \leq 1 A(n, k) \rightarrow \exists e \forall n \left[ {e} (n) \downarrow \land A(n, {e}(n)) \right]$$

with

$$A(n, k) \equiv \forall x P(x, k) \lor \forall y Q(y, k),$$

where P, Q are primitive recursive.

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V. Lifschitz’s idea

\[ D_e = \{ k \leq (e)^0 | (e)^0 \uparrow \} \]

Fact 1: There is a partial recursive function \( \wp \) such that \( D_e \) is a singleton \( \Leftrightarrow \wp (e)^0 \downarrow \land \wp (e)^0 \in D_e \)

Fact 2: There is no index \( g \) of a partial recursive function such that, for all \( e \), \( D_e \neq \emptyset \Leftrightarrow g \cdot e \downarrow \land g \cdot e \in D_e \)
V. Lifschitz’s idea

Let

\[ D_e = \{ k \leq (e)_1 \mid (e)_0 \cdot k \uparrow \} \]
V. Lifschitz’s idea

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\[ D_e = \{ k \leq (e)_1 \mid (e)_0 \bullet k \uparrow \} \]

**Fact 1**: There is a partial recursive function \( \varphi \) such that

\[ D_e \text{ is a singleton} \Rightarrow \varphi(e) \downarrow \land \varphi(e) \in D_e \]
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- Let

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- **Fact 1:** There is a partial recursive function \( \varphi \) such that

\[ D_e \text{ is a singleton } \Rightarrow \varphi(e) \downarrow \land \varphi(e) \in D_e \]

- **Fact 2:** There is no index \( g \) of a partial recursive function such that, for all \( e \),

\[ D_e \neq \emptyset \Rightarrow g \cdot e \downarrow \land g \cdot e \in D_e \]
• **Fact 2**: There is no index $g$ of a partial recursive function such that, for all $e$,

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$$D_e \neq \emptyset \Rightarrow g \cdot e \downarrow \land g \cdot e \in D_e$$

• **Proof of Fact 2:** Assume otherwise. Take two disjoint, recursively inseparable r.e. sets $W_f$ and $W_h$. One can find a recursive function $F$ such that

$$\forall n[F(n) \cdot 0 \simeq f \cdot n \land F(n) \cdot 1 \simeq h \cdot n].$$

Then always $D_{\langle F(n),1 \rangle} \neq \emptyset$, so $g \cdot \langle F(n), 1 \rangle \in D_{\langle F(n),1 \rangle}$. Let $X := \{n \mid g \cdot \langle F(n), 1 \rangle = 0\}$. Then $X \cap W_f = \emptyset$ and $W_h \subseteq X$. So we have found a recursive separation between $W_f$ and $W_h$. Contradiction!
Lifschitz realizability

• $e \vdash LA$ iff $A$ is true for atomic $A$.

• $e \vdash LA \land LB$ iff ($e_0 \vdash LA$ and ($e_1 \vdash LB$).

• $e \vdash LA \rightarrow LB$ iff $\forall d \in \mathbb{N} \left[ d \vdash LA \rightarrow e \downarrow \land e \downarrow d \vdash LB \right]$.

• $e \vdash LA \forall x F(x)$ iff for all $n \in \mathbb{N}$, $e \downarrow n \land e \downarrow n \vdash LF(n)$.

• $e \vdash LA \exists x F(x)$ iff $D \neq \emptyset$ and for every $d \in D$, $(d \vdash LF((d)_0))$. 

LIFSCHITZ REALIZABILITY FOR SET THEORIES
Lifschitz realizability

• $e \vdash_L A$ iff $A$ is true for atomic $A$. 
Lifschitz realizability

- $e \models_L A$ iff $A$ is true for atomic $A$.
- $e \models_L A \land B$ iff $(e)_0 \models_L A$ and $(e)_1 \models_L B$
Lifschitz realizability

- $e \models_L A$ iff $A$ is true for atomic $A$.
- $e \models_L A \land B$ iff $(e)_0 \models_L A$ and $(e)_1 \models_L B$
- $e \models_L A \rightarrow B$ iff $\forall d \in \mathbb{N}[d \models_L A \rightarrow e \bullet d \downarrow \land e \bullet d \models_L B]$
Lifschitz realizability

\begin{itemize}
\item $e \vDash_L A$ iff $A$ is true for atomic $A$.
\item $e \vDash_L A \land B$ iff $(e)_0 \vDash_L A$ and $(e)_1 \vDash_L B$
\item $e \vDash_L A \rightarrow B$ iff $\forall d \in \mathbb{N}[d \vDash_L A \rightarrow e \circ d \downarrow \land e \circ d \vDash_L B$
\item $e \vDash_L \forall x F(x)$ iff for all $n \in \mathbb{N}$, $e \circ n \downarrow \land e \circ n \vDash_L F(n)$
\end{itemize}
Lifschitz realizability

- $e \models_L A$ iff $A$ is true for atomic $A$.
- $e \models_L A \land B$ iff $(e)_0 \models_L A$ and $(e)_1 \models_L B$
- $e \models_L A \rightarrow B$ iff $\forall d \in \mathbb{N}[d \models_L A \rightarrow e \bullet d \downarrow \land e \bullet d \models_L B$
- $e \models_L \forall x F(x)$ iff for all $n \in \mathbb{N}, e \bullet n \downarrow \land e \bullet n \models_L F(n)$
- $e \models_L \exists x F(x)$ iff $D_e \neq \emptyset$ and for every $d \in D_e$, $(d)_1 \models_L F((d)_0)$. 
Definition:

Let $a, b \in V(K_1)$ and $e \in \mathbb{N}$.

- $e \models_L \phi \land \psi$ iff $(e)_0 \models_L \phi \land (e)_1 \models_L \psi$
- $e \models_L \neg \phi$ iff $\forall f \in \mathbb{N} \neg f \models_L \phi$
- $e \models_L \phi \rightarrow \psi$ iff $\forall f \in \mathbb{N} \left[ f \models_L \phi \rightarrow e \circ f \models_L \psi \right]$
- $e \models_L \forall x \phi$ iff $\forall c \in V(K_1) e \models_L \phi[x/c]$
- $e \models_L \exists x \phi$ iff $D_e \neq \emptyset \land \forall d \in D_e \exists c \in V(K_1) d \models_L \phi[x/c]$
The atomic cases

**Definition:**

\[ e \models_L a \in b \quad \text{iff} \quad D_e \neq \emptyset \land \forall d \in D_e \exists c \left[ \langle (d)_0, c \rangle \in b \land (d)_1 \models_L a = c \right] \]

\[ e \models_L a = b \quad \text{iff} \quad \forall f, x \left[ \left( \langle f, x \rangle \in a \rightarrow (e)_0 \bullet f \models_L x \in b \right) \land \left( \langle f, x \rangle \in b \rightarrow (e)_1 \bullet f \models_L x \in a \right) \right] \]
The atomic cases

**Definition:**

\[ e \models_L a \in b \text{ iff } D_e \neq \emptyset \land \forall d \in D_e \exists c \left[ \langle (d)_0, c \rangle \in b \land (d)_1 \models_L a = c \right] \]

\[ e \models_L a = b \text{ iff } \forall f, x \left[ (\langle f, x \rangle \in a \to (e)_0 \cdot f \models_L x \in b) \land (\langle f, x \rangle \in b \to (e)_1 \cdot f \models_L x \in a) \right] \]

**Corollary:**

**Bounded Quantifiers**

\[ e \models_L \forall x \in a \phi \text{ iff } \forall \langle f, c \rangle \in a e \cdot f \models_L \phi[x/c] \]

\[ e \models_L \exists x \in a \phi \text{ iff } D_e \neq \emptyset \land \forall d \in D_e \exists c \left[ \langle (d)_0, c \rangle \in a \land (d)_1 \models_L \phi[x/c] \right] \]
A variant of Lifschitz realizability for Set Theory

**Definition:**

Let $a, b \in V(K_1)$ and $e \in \mathbb{N}$.

- $e \models_{L_v} \phi \land \psi$ iff $D_e \neq \emptyset \land \forall d \in D_e [(d)_0 \models_{L_v} \phi \land (d)_1 \models_{L_v} \psi]$

- $e \models_{L_v} \phi \rightarrow \psi$ iff $D_e \neq \emptyset \land \forall d \in D_e \forall f \in \mathbb{N} [f \models_{L_v} \phi \rightarrow d \circ f \models_{L_v} \psi]$

- $e \models_{L_v} \forall x \phi$ iff $D_e \neq \emptyset \land \forall d \in D_e \forall c \in V(K_1) d \models_{L_v} \phi[x/c]$

- $e \models_{L_v} \exists x \phi$ iff $D_e \neq \emptyset \land \forall d \in D_e \exists c \in V(K_1) d \models_{L_v} \phi[x/c]$
The atomic cases

**Definition:**

$e \models_{L_v} a \in b \iff D_e \neq \emptyset \land$

$\forall d \in D_e \exists c \left[ \langle d,0 \rangle, c \rangle \in b \land (d)_1 \models_{L_v} a = c \right]$

$e \models_{L_v} a = b \iff D_e \neq \emptyset \land$

$\forall d \in D_e \forall f, x \left[ (\langle f, x \rangle \in a \rightarrow (d)_0 \bullet f \models_{L_v} x \in b) \land (\langle f, x \rangle \in b \rightarrow (d)_1 \bullet f \models_{L_v} x \in a) \right]$
Theorem: (Chen, R)

Lifschitz realizability is sound for $\text{IZF} + \text{Markov's principle}$, i.e. $\forall k_1 \in \text{L}_{\text{IZF}} + \text{Markov's principle}$.

Moreover, $\forall k_1 \in \text{L}_{\text{CT}_{0}}$ but $\forall k_1 \notin \text{L}_{\text{CT}_{0}}$.

Hence $\forall k_1 \in \text{L}_{\forall f \in \omega \omega \text{f is a computable function}}$.

Hence $\forall k_1 \notin \text{L}_{\text{AC}_{\omega, 2}}$.
Theorem: (Chen, R)

- Lifschitz realizability is sound for $\text{IZF} + \text{Markov’s principle}$, i.e $V(K_1) \models L \text{IZF} + \text{Markov’s principle}$.

- Hence $V(K_1) \models L \forall f \in \omega \omega$ if $f$ is a computable function.

- Hence $V(K_1) \not\models L \exists \omega, 2$. 

\textsc{Lifschitz Realizability for Set Theories}
Theorem: (Chen, R)

- Lifschitz realizability is sound for $\text{IZF} + \text{Markov’s principle}$, i.e. $V(K_1) \models_L \text{IZF} + \text{Markov’s principle}$.
- Moreover, $V(K_1) \models_L \text{CT}_0$ but $V(K_1) \not\models_L \text{CT}_0$. 

Theorem: (Chen, R)

• Lifschitz realizability is sound for $\text{IZF} + \text{Markov’s principle}$, i.e $V(K_1) \models_L \text{IZF} + \text{Markov’s principle}$.

• Moreover, $V(K_1) \models_L \text{CT}_0$ but $V(K_1) \not\models_L \text{CT}_0$.

• Hence $V(K_1) \models_L \forall f \in \omega \omega f$ is a computable function.
Theorem: (Chen, R)

- Lifschitz realizability is sound for $\text{IZF} + \text{Markov’s principle}$, i.e $V(K_1) \models_L \text{IZF} + \text{Markov’s principle}$.
- Moreover, $V(K_1) \models_L \text{CT}_0$! but $V(K_1) \not\models_L \text{CT}_0$.
- Hence $V(K_1) \models_L \forall f \in \omega\omega f$ is a computable function.
- Hence $V(K_1) \not\models_L \text{AC}_{\omega,2}$. 
The constructive Dedekind reals

**Definition:**

A **Dedekind cut** is a pair \((L, U)\) of subsets of \(\mathbb{Q}\), satisfying:

1. \(q \in L \iff \exists r \in L \ q < r\)
2. \(q \in U \iff \exists r \in U \ r < q\)
3. \(q \in L \land r \in U \implies q < r\)
4. \(\forall n \exists q \in L \exists r \in U \ r - q < 1/n\) (locatedness)

- Call \((L, U)\) a strong real if there exists \(f : \mathbb{Q}^2 \to \mathbb{N}\) such that:
  - \(\forall q, r \ (q < r \implies [f(q, r) = 0 \land q \in L] \lor [f(q, r) \neq 0 \land r \in U])\)
The constructive Dedekind reals

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\[ q < r \rightarrow \left[ f(q, r) = 0 \land q \in L \right] \lor \left[ f(q, r) \neq 0 \land r \in U \right] \]
A **Dedekind cut** is a pair \((L, U)\) of subsets of \(\mathbb{Q}\), satisfying:

\[(i) \quad q \in L \leftrightarrow \exists r \in L \quad q < r \]
\[(ii) \quad q \in U \leftrightarrow \exists r \in U \quad r < q \]
The constructive Dedekind reals

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3. \(q \in L \land r \in U \rightarrow q < r\)

\(\text{locatedness}\)
The constructive Dedekind reals

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2. \(q \in U \iff \exists r \in U \ r < q\)  
3. \(q \in L \land r \in U \rightarrow q < r\)  
4. \(\forall n \exists q \in L \exists r \in U \ r - q < \frac{1}{n}\) (*locatedness*)
**The constructive Dedekind reals**

**Definition:**

A **Dedekind cut** is a pair \((L, U)\) of subsets of \(\mathbb{Q}\), satisfying:

\[
(i) \quad q \in L \iff \exists r \in L \ q < r \\
(ii) \quad q \in U \iff \exists r \in U \ r < q \\
(iii) \quad q \in L \land r \in U \to q < r \\
(iv) \quad \forall n \exists q \in L \exists r \in U \ r - q < \frac{1}{n} \text{ (locatedness)}
\]

- **Call** \((L, U)\) a **strong real** if there exists \(f : \mathbb{Q}^2 \to \mathbb{N}\) such that

\[
\forall q, r \ (q < r \to [[f(q, r) = 0 \land q \in L] \lor [f(q, r) \neq 0 \land r \in U]]).
\]
Van Oosten investigated the Dedekind reals in the so-called \textbf{Lifschitz topos}.

\textit{Theorem: (van Oosten)}

\textit{The Dedekind reals and the Cauchy reals coincide in the Lifschitz topos.}
Van Oosten investigated the Dedekind reals in the so-called **Lifschitz topos**.

**Theorem: (van Oosten)**

The Dedekind reals and the Cauchy reals coincide in the Lifschitz topos.

This result uses

**Theorem: (e.g. Troelstra, van Dalen 5.5.10)**

The collection of Cauchy reals is order-isomorphic to the collection of strong reals.
The Dedekind reals in $V(K_1)_L$

**Theorem:**

$V(K_1) \models_L \mathbb{R}^d \cong \mathbb{R}$. 
Metatheory required for Lifschitz realizability?

Van Oosten showed that Lifschitz realizability for HA can be dealt with in HA + MP + PR + B_Σ⁰₂ - MP.

MP PR is ¬¬∃x P(x) → ∃P(x) for P primitive recursive.

B_Σ⁰₂ - MP is ¬¬∃x ≤ y ∀n Q(x, n, e) → ∃x ≤ y ∀n Q(x, n, e) for Q primitive recursive.
Van Oosten showed that Lifschitz realizability for $\text{HA}$ can be dealt with in

$$\text{HA} + \text{MP}_{PR} + \text{B}\Sigma_2^0\text{-MP}$$
Van Oosten showed that Lifschitz realizability for HA can be dealt with in

\[ \text{HA} + \text{MP}_{PR} + \text{B}^{\Sigma_2^0}\text{-MP} \]

- **MP\text{\_PR}** is

  \[ \neg \neg \exists x P(x) \rightarrow \exists P(x) \]

for \( P \) primitive recursive.
Van Oosten showed that Lifschitz realizability for $\text{HA}$ can be dealt with in

$$\text{HA} + \text{MP}_{PR} + \text{B}\Sigma^0_2\text{-MP}$$

**MP$_{PR}$** is

$$\neg\neg \exists x P(x) \rightarrow \exists P(x)$$

for $P$ primitive recursive.

**B$\Sigma^0_2$-MP** is

$$\neg\neg \exists x \leq y \forall n Q(x, n, e) \rightarrow \exists x \leq y \forall n Q(x, n, e)$$

for $Q$ primitive recursive.
Can Lifschitz realizability be used for other PCA's?

Yes! Take e.g. the PCA where "\(e \cdot n\)" means "apply the \(e\)th hyperarithmetic function to \(n\)."

Theorem: IZF + Excluded Middle for arithmetic formulas \(\not\vdash AC_{\omega, 2}\).
Can Lifschitz realizability be used for other PCA’s?

- Yes! Take e.g. the PCA where “$e \bullet n$" means “apply the $e^{th}$ hyperarithmetic function to $n$".

Theorem: $\text{IZF} + \text{Excluded Middle for arithmetic formulas} \not\vdash AC_\omega, 2$. 
Can Lifschitz realizability be used for other PCA’s?

- Yes! Take e.g. the PCA where “$e \bullet n$” means “apply the $e^{th}$ hyperarithmetic function to $n$”.

**Theorem:**

$\text{IZF} + \text{Excluded Middle for arithmetic formulas} \not\vdash \text{AC}_{\omega,2}$. 
The End
The End

Thank you!
Definition:

Let $T$ be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant $\omega$ denoting the set of von Neumann natural numbers and for each $n$ a constant $\bar{n}$ denoting the $n$-th element of $\omega$.

- $T$ has the disjunction property, $\text{DP}$, if whenever

$$T \vdash \psi \lor \theta$$

holds for sentences $\psi$ and $\theta$ of $T$, then

$$T \vdash \psi \quad \text{or} \quad T \vdash \theta.$$
• $T$ has the \textit{numerical existence property}, \textbf{NEP}, if whenever

$$T \vdash (\exists x \in \omega) \phi(x)$$

holds for a formula $\phi(x)$ with at most the free variable $x$, then

$$T \vdash \phi(\bar{n})$$

for some $n$.

• $T$ has the \textit{existence property}, \textbf{EP}, if whenever

$$T \vdash \exists x \phi(x)$$

holds for a formula $\phi(x)$ having at most the free variable $x$, then there is a formula $\vartheta(x)$ with exactly $x$ free, so that

$$T \vdash \exists! x [\vartheta(x) \land \phi(x)].$$
• $T$ has the weak existence property, $\text{wEP}$, if whenever

$$T \vdash \exists x \phi(x)$$

holds for a formula $\phi(x)$ having at most the free variable $x$, then there is a formula $\vartheta(x)$ with exactly $x$ free, so that

$$T \vdash \exists ! x \vartheta(x),$$

$$T \vdash \forall x [\vartheta(x) \rightarrow \exists u \ u \in x],$$

$$T \vdash \forall x [\vartheta(x) \rightarrow \forall u \in x \phi(x)].$$
• $T$ is closed under *Church’s rule*, $\text{CR}$, if whenever

$$T \vdash (\forall x \in \omega)(\exists y \in \omega) \phi(x, y)$$

holds for some formula of $T$ with at most the free variables shown, then for some number $e$,

$$T \vdash (\forall x \in \omega)\phi(x, \{\bar{e}\}(x)),$$

where $\{e\}(x)$ stands for the result of applying the $e$-th partial recursive function to $x$.

• Let $f : \omega \rightarrow \omega$ convey that $f$ is a function from $\omega$ to $\omega$. $T$ is closed under the variant of *Church’s rule*, $\text{CR}_1$, if whenever

$$T \vdash \exists f \left[ f : \omega \rightarrow \omega \land \psi(f) \right]$$

(with $\psi(f)$ having no variables but $f$), then for some number $e$,

$$T \vdash (\forall x \in \omega)(\exists y \in \omega)(\{\bar{e}\}(x) = y) \land \psi(\{\bar{e}\}).$$
Slightly abusing terminology, we shall also say that $T$ enjoys any of these properties if this holds only for a definitional extension of $T$ rather than $T$. 
Let $IZF_R$ result from $IZF$ by replacing Collection with Replacement, and let $CST$ be Myhill’s constructive set theory. Also let $CST^-$ be $CST$ without the axioms of countable and dependent choice.

**Theorem:**

(i) (Myhill) $IZF_R$ and $CST^-$ have the DP, NEP, and the EP. 
$CST$ has the DP and NEP.

(ii) (Beeson) $IZF$ has the DP and the NEP.

(iii) (Friedman, Scedrov) $IZF$ does not have the EP.

Ignoring the trivial counterexamples, classical theories never have the DP. But classical set theories can have the EP. E.g., an extension of $ZF$ has the EP if and only if it proves that all sets are ordinal definable, i.e., $V = OD$. 

**Lifschezit Realizability for Set Theories**
If $a$ is an ordered pair, i.e., $a = \langle x, y \rangle$ for some sets $x, y$, then we use $1^{st}(a)$ and $2^{nd}(a)$ to denote the first and second projection of $a$, respectively; that is, $1^{st}(a) = x$ and $2^{nd}(a) = y$. For a class $X$ we denote by $\mathcal{P}(X)$ the class of all sets $y$ such that $y \subseteq X$. 
The general realizability structure

Definition:

\[ \text{COV}(a, b) \quad \text{iff} \quad (\forall x \in b) 1^{st}(2^{nd}(x)) \in a. \]

\[
\begin{align*}
V^*_\alpha &= \bigcup_{\beta \in \alpha} \{ \langle a, b \rangle : a \in V_\beta; b \subseteq \omega \times V^*_\beta; \text{COV}(a, b) \} \\
V_\alpha &= \bigcup_{\beta \in \alpha} \mathcal{P}(V_\beta) \\
V^* &= \bigcup_{\alpha} V^*_\alpha \\
V &= \bigcup_{\alpha} V_\alpha.
\end{align*}
\]
The definition of $V^*_\alpha$ in (4) is perhaps a bit involved. Note first that all the elements of $V^*$ are ordered pairs $\langle a, b \rangle$ such that $b \subseteq \omega \times V^*$. For an ordered pair $\langle a, b \rangle$ to enter $V^*_\alpha$ the first conditions to be met are that $a \in V_\beta$ and $b \subseteq \omega \times V^*_\beta$ for some $\beta \in \alpha$. Furthermore, it is required that $a$ contains enough elements from the transitive closure of $b$ in that whenever $\langle e, c \rangle \in b$ then $1^{st}(c) \in a$. 
Lemma:

(CZF).

(i) V and $V^*$ are cumulative: for $\beta \in \alpha$, $V_\beta \subseteq V_\alpha$ and $V^*_\beta \subseteq V^*_\alpha$.

(ii) For all sets $a, a \in V$.

(iii) If $a, b$ are sets, $b \subseteq \omega \times V^*$ and $\text{COV}(a, b)$, then $\langle a, b \rangle \in V^*$. 
We now proceed to define a notion of extensional realizability over $V^*$. We use lower case gothic letters

$$a, b, c, d, e, f, g, h, n, m, p, q \ldots$$

as variables to range over elements of $V^*$ while variables $e, c, d, f, g, \ldots$ will be reserved for elements of $\omega$. Each element $a$ of $V^*$ is an ordered pair $\langle x, y \rangle$, where $x \in V$ and $y \subseteq \omega \times V^*$; and we define the components of $a$ by

$$a^\circ := 1^{st}(a) = x$$

$$a^* := 2^{nd}(a) = y.$$ 

**Lemma:**

*For every $a \in V^*$, if $\langle e, c \rangle \in a^*$ then $c^\circ \in a^\circ$.***

**Proof:** This is immediate by the definition of $V^*$. $\square$

If $\varphi$ is a sentence with parameters in $V^*$, then $\varphi^\circ$ denotes the formula obtained from $\varphi$ by replacing each parameter $a$ in $\varphi$ with $a^\circ$. 
Defining realizability

**Definition:**

Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers.

We define $e \vDash_t \phi$ for sentences $\phi$ with parameters in $V^*$.

\[
e \vDash_t a \in b \iff a^\circ \in b^\circ \land \exists c \left[ \langle (e)_0, c \rangle \in b^* \land (e)_1 \vDash_t a = c \right]
\]

\[
e \vDash_t a = b \iff a^\circ = b^\circ \land \forall f \forall c \left[ \langle f, c \rangle \in a^* \rightarrow (e)_0 f \vDash_t c \in b \right] \land \forall f \forall c \left[ \langle f, c \rangle \in b^* \rightarrow (e)_0 f \vDash_t c \in a \right]
\]

\[
e \vDash_t \phi \land \psi \iff (e)_0 \vDash_t \phi \land (e)_1 \vDash_t \psi
\]
Definition:

\[ e \models_t \neg \phi \iff \neg \phi^o \land \forall f \neg f \models_t \phi \]

\[ e \models_t \phi \rightarrow \psi \iff (\phi^o \rightarrow \psi^o) \land \forall f \left[ f \models_t \phi \rightarrow ef \models_t \psi \right] \]

\[ e \models_t (\forall x \in a) \phi \iff (\forall x \in a^o)\phi^o \land \\
\forall f \forall b (\langle f, b \rangle \in a^* \rightarrow ef \models_t \phi[x/b]) \]

\[ e \models_t (\exists x \in a) \phi \iff \exists b \left( \langle (e)_0, b \rangle \in a^* \land (e)_1 \models_t \phi[x/b] \right) \]

\[ e \models_t \forall x \phi \iff \forall a \ e \models_t \phi[x/a] \]

\[ e \models_t \exists x \phi \iff \exists a \ e \models_t \phi[x/a] \]
Definition:

By $\in$-recursion we define for every set $x$ a set $x^{st}$ as follows:

$$x^{st} = \langle x, \{\langle 0, u^{st} \rangle : u \in x \} \rangle.$$ (4)

Lemma:

For all sets $x$, $x^{st} \in V^*$ and $(x^{st})^\circ = x$.

Lemma:

If $e \models_t \phi$ then $\phi^\circ$. 
Our hopes for showing DP and NEP for CZF and related systems rest on the following results.

**Lemma:**

If $e \vdash t (\exists x \in a) \phi$ then

$$\exists b \left( ((e)_0, b) \in a^* \land \phi^*[x/b^*] \right).$$

**Lemma:**

If $e \vdash t \phi \lor \psi$ then

$$[ (e)_0 = 0 \land \phi^* ] \lor [ (e)_0 \neq 0 \land \psi^* ].$$
Main Results

The main metamathematical result obtained via this tool are the following.

**Theorem:**

*The DP and the NEP hold true for CZF and CZF + REA. Both theories are closed under CR and CR₁, too. One can also add the following choice principles: ACω, DC, RDC, PA.*