σ-continuity and related forcings

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Idealized forcings

Many classical forcing notions arise as quotient Boolean algebras of $\mathrm{Bor}(X)$ modulo an ideal I in a Polish space. We denote them by $\mathbb{P}_I$. E.g. Cohen forcing is associated with meager sets, random forcing with null sets, Sacks forcing with countable sets and Miller forcing with K_σ sets in the Baire space.

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and their tree representations

And very often these forcings can be equivalently described as forcings with certain families of trees. In the examples above Sacks forcing is the forcing with perfect trees and Miller forcing with superperfect subtrees of $\omega^{<\omega}$.

Names for reals

Very often idealized forcings $\mathbb{P}_I$ offer convenient names for reals in the extension. One is $\dot{g}$, which names the generic real. And usually all other reals in the extension are of the form $f(\dot{g})$ where $f : \omega^\omega \to \omega^\omega$ is some function from the ground model.

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Theorem (Zapletal)

If the forcing notion $\mathbb{P}_I$ is proper and $\dot{x}$ is a name for a real then for each $B \in \mathbb{P}_I$ there is a condition $C \leq B$ and a Borel function $f : C \to \omega^\omega$ such that

$$C \Vdash \dot{x} = f(\dot{g}).$$

Continuous reading of names

We say that a forcing notion $\mathbb{P}_I$ has *continuous reading of names* if for any condition $B \in \mathbb{P}_I$ and a name for a real there is $C \leq B$ and a continuous function $f : C \to \omega^\omega$ such that $C \Vdash \dot{x} = f(\dot{g})$. Keep in mind, however, that this (at least formally) depends on the topology of the space.

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Remark

If a forcing $\mathbb{P}_I$ is proper then continuous reading of names translates to a topological property: for each Borel I-positive set B and a Borel function $f : B \rightarrow \omega^\omega$ there is an I-positive set $C \subseteq B$ such that $f \restriction C$ is continuous. Most classical forcings (Cohen, random, Miller, etc.) have this property.

One old problem

Recall an old question of Lusin whether there exists a Borel function which is not σ-continuous, i.e. its domain X cannot be decomposed into countably many sets X_n such that on each X_n the function is continuous.

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The function P

An example of such a function has been given by Pawlikowski who defined $P : (\omega+1)^\omega \to \omega^\omega$ in the following way:

$$P(x)(n) = \begin{cases} x(n)+1 & \text{if } x(n) < \omega \\ 0 & \text{if } x(n) = \omega \end{cases}$$

The Steprāns forcing

The function P naturally gives rise to an ideal I_P on $(\omega+1)^\omega$, namely the family of sets on which P is σ-continuous. This ideal has Borel base by the Kuratowski extension theorem. The forcing notion $\mathbb{P}_{I_P}$ is called Steprāns forcing.

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Note that Steprāns forcing is an example of an idealized forcing without the property of continuous reading of names in the natural topology of the space $(\omega+1)^\omega$. The function P itself is a Borel function which is not continuous on any I_P-positive set. Besides, Steprāns forcing is proper. Actually, it can be shown that it satisfies Baumgartner's Axiom A.

And what about changing the topology?

Having one Borel function we can always change the topology without changing the Borel structure so that this function becomes continuous. In case of the function P this procedure gives the Baire space topology. And in this topology the function P is no longer an obstacle for continuous reading of names

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Theorem (Zapletal)

If the ideal I is generated by closed sets in a topology $\mathcal{T}$ and the forcing $\mathbb{P}_I$ is proper then the forcing $\mathbb{P}_I$ has continuous reading of names (in the topology $\mathcal{T}$).

Questions

There are three questions that come to mind:

1. Is the ideal I_P generated by closed sets in the Baire topology of $(\omega+1)^\omega$?
2. Does Stepāns forcing have continuous reading of names in the Baire topology?
3. If it does, then are any forcings $\mathbb{P}_I$ which are proper and do not have continuous reading of names in any Polish topology (which gives the same Borel structure)?

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Note

The third question was originally posed in a paper of Hrusak and Zapletal. And the first question appeared in the original paper of Steprāns.

Generating by closed sets

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Sketch of the proof

Let us first consider the following set $A = \{\alpha_n, \beta_n : n < \omega\} \subseteq (\omega+1)^\omega$, where

$$\alpha_n(0) = n, \quad \alpha_n(k) = \omega \text{ for } k > 0$$

$$\beta_n(n) = 0, \quad \beta_n(k) = \omega \text{ for } k \neq n.$$

Note that $P{\restriction}A$ is continuous. On the other hand, $P{\restriction}\overline{A}$ is not continuous since $\alpha = (\omega, \omega, \dots) \in \overline{A}$ and $\alpha_n \to \alpha$, whereas $P(\alpha_n) \not\to P(\alpha)$.

Identify $(\omega+1)^\omega \simeq ((\omega+1)^\omega)^\omega$. Now P becomes P^ω. First note that P^ω is continuous on A^ω as a product of continuous functions, so $A^\omega \in I_{P^\omega}$. So let us prove that A^ω cannot be covered by countably many sets $F_n, n<\omega$ closed in the Baire topology, with each $F_n \in I_{P^\omega}$.

Suppose that $A^\omega \subseteq \bigcup_n F_n$ and F_n are closed in the Baire topology. As A is a discrete set in $(\omega+1)^\omega$ (in both topologies), the relative topology (with respect to any of these two) on A^ω is that of the Baire space. $F_n \cap A^\omega$ are relatively closed, so by the Baire category theorem, one of them has nonempty interior. This means that there is $n<\omega$, $k<\omega$ and $\alpha \in A^k$ such that $\alpha^\frown A^{\omega\setminus k} \subseteq F_n$. Without loss of generality $k=0$ and $\overline{A^\omega} \subseteq F_n$. But $\overline{A^\omega} = (\overline{A})^\omega$ and $\overline{A}$ contains a convergent sequence $\alpha_n \to \alpha$ such that $P(\alpha_n) \not\to P(\alpha)$. So if $A' = \{\alpha, \alpha_n : n<\omega\}$ then $P[A']$ is a discrete set and $(A')^\omega \subseteq F_n$. Notice, however, that $P^\omega\restriction(A')^\omega = (P\restriction A')^\omega$ is not σ-continuous, since it can obviously be factorized by P. Hence $F_n \notin I_{P^\omega}$, which ends the proof.

Continuous reading of names

Nevertheless, there are also other reasons for continuous reading of names. And Steprāns forcing has this property.

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Why?

The proof becomes easy once we establish another description of the forcing in terms of trees.

Definition

Let T be a subtree of $(\omega+1)^{<\omega}$. T is a *wide tree* if for each $\tau \in T$ there is an extension $\tau \subseteq \sigma \in T$ such that:

- $\sigma^\frown\omega \in T$,
- $\lim T \cap [\sigma^\frown\omega]$ is nowhere dense in $\lim T$.

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Consequences

The forcing with wide trees is much easier to deal with than the original $\mathbb{P}_{I_P}$. One can deduce from it continuous reading of names, Axiom A and many other properties.

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As Steprāns forcing turns out to have continuous reading of names in the extended topology, the third question becomes reasonable.

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Proposition

There are forcings $\mathbb{P}_I$ which are proper and do not have continuous reading of names in any Polish topology (which gives the same Borel structure)

First notice that any presentation of a Polish space X is given by a Borel isomorphism with another Polish space Y and the latter can be assumed to be a G_δ subset of $[0,1]^\omega$. Instead of ω^ω let us consider $X = (\omega^\omega)^2$ with its product topology. Note that each G_δ set in $[0,1]^\omega$ as well as a Borel isomorpshism from ω^ω to X can be coded by a real. Let $x \in \omega^\omega$ code a pair (G_x, f_x) that defines a presentation of X as above, i.e. $f_x : G_x \to X$. For $x \in \omega^\omega$ $f_x^{-1}[X_x]$ (X_x denotes the vertical section of X at x) is an uncountable Borel set in G_x and contains a copy C_x of $(\omega+1)^\omega$. Let I_x be the transported ideal I_P from C_x to X_x. We define an ideal I on $\mathrm{Bor}(X)$ as follows:

$$I = \{A \in \mathrm{Bor}(X) : \forall x \in \omega^\omega \, A_x \in I_x\}.$$

$\mathbb{P}_I$ does not have continuous reading of names in any presentation for if (G_x, f_x) defines a presentation then $(P \circ f_x^{-1}){\restriction}(f_x[C_x])$ is a counterexample to continuous reading of names in its topology.

Thank You!