Algebraic dynamics and definable sets

Thomas Scanlon

UC Berkeley

4 July 2008

What we mean by dynamics

Dynamics understood as the study of processes which evolve over time according to some iterative rule has been an essential part of mathematics for millenia.

What we mean by dynamics

Dynamics understood as the study of processes which evolve over time according to some iterative rule has been an essential part of mathematics for millenia and has been studied via methods in logic for almost as long.

What we mean by dynamics

Dynamics understood as the study of processes which evolve over time according to some iterative rule has been an essential part of mathematics for millenia and has been studied via methods in logic for almost as long.

We shall concentrate on discrete dynamics, and more specifically, on issues around the iteration of a single function, and even more specifically (mostly) with algebraic dynamics and even there only on a small fragment of the theory.

Dynamics as iteration

For us, a dynamical system is given by a self-map $f : X \to X$ from a set X back to itself.

Dynamics as iteration

For us, a dynamical system is given by a self-map $f : X \to X$ from a set X back to itself.

The dynamics of a dynamical system are really understood in terms of the properties of $f : X \to X$ under iteration, namely, of the associated function

$$F : \mathbb{N} \times X \to X$$

$$(n, x) \mapsto f^n(x)$$

Dynamics as iteration

For us, a dynamical system is given by a self-map $f : X \to X$ from a set X back to itself.

The dynamics of a dynamical system are really understood in terms of the properties of $f : X \to X$ under iteration, namely, of the associated function

$$F : \mathbb{N} \times X \to X$$

$$(n, x) \mapsto f^n(x)$$

Indeed, in the literature on dynamical systems, the function F, possibly with $\mathbb{N}$ replaced by another semigroup and assumed to satisfy a cocycle condition reflecting the iterative nature of the dynamics, is taken as primitive.

For the most part, we shall specialize to the case of algebraic dynamics where $f : X \to X$ is a regular map of algebraic varieties.

Algebraic dynamics

For the most part, we shall specialize to the case of algebraic dynamics where $f : X \to X$ is a regular map of algebraic varieties.
Fix an algebraically closed field K (which you could take to be $\mathbb{C}$ without much loss of generality).

Definition

An algebraic variety (really, an affine algebraic subvariety of $\mathbb{A}^n_K$) is a set of the form

$$X = \{(a_1, \ldots, a_n) \in K^n : \bigwedge G_j(\mathbf{a}) = 0\}$$

where each $G_j \in K[x_1, \ldots, x_n]$ is a polynomial in n variables.

Algebraic dynamics

For the most part, we shall specialize to the case of algebraic dynamics where $f : X \to X$ is a regular map of algebraic varieties.
Fix an algebraically closed field K (which you could take to be $\mathbb{C}$ without much loss of generality).

Definition

An algebraic variety (really, an affine algebraic subvariety of $\mathbb{A}^n_K$) is a set of the form

$$X = \{(a_1, \ldots, a_n) \in K^n : \bigwedge G_j(\mathbf{a}) = 0\}$$

where each $G_j \in K[x_1, \ldots, x_n]$ is a polynomial in n variables.

Definition

A regular function $f : X \to X$ is a function of the form

$$(x_1, \ldots, x_n) \mapsto (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$$

where each F_i is a polynomial.

Observations about decidability and algebraic dynamics

- If $f : X \to X$ is an algebraic dynamical system over K, then X and f may be regarded as K-definable sets. Indeed, by the Tarski-Chevallay quantifier elimination theorem, every definable set is a finite Boolean combination of varieties.) As the theory of algebraically closed fields is decidable, questions about $f : X \to X$ expressible in the language of rings may be algorithmically resolved.
- The associated function $F : \mathbb{N} \times X \to X$ expressing the iteration of f makes essential reference to $\mathbb{N}$ so that in general one might expect questions about dynamical properties of $f : X \to X$ to be intractable.

- If $f : X \to X$ is an algebraic dynamical system over K, then X and f may be regarded as K-definable sets. As the theory of algebraically closed fields is decidable, questions about $f : X \to X$ expressible in the language of rings may be algorithmically resolved.
- The associated function $F : \mathbb{N} \times X \to X$ expressing the iteration of f makes essential reference to $\mathbb{N}$ so that in general one might expect questions about dynamical properties of $f : X \to X$ to be intractable.

- If $f : X \to X$ is an algebraic dynamical system over K, then X and f may be regarded as K-definable sets. As the theory of algebraically closed fields is decidable, questions about $f : X \to X$ expressible in the language of rings may be algorithmically resolved.
- The associated function $F : \mathbb{N} \times X \to X$ expressing the iteration of f makes essential reference to $\mathbb{N}$ so that in general one might expect questions about dynamical properties of $f : X \to X$ to be intractable.

It is very easy to find natural problems about algebraic dynamical systems to which arbitrarily complicated problems in the theory of arithmetic are reducible. The real task is to isolate a class of algebraic dynamics controlled by tame geometries in the model theoretic sense.

Periodic points

Definition

Given a dynamical system $f : X \to X$ we say that a point $a \in X$ is periodic if $f^n(a) = a$ for some $n \in \mathbb{Z}_+$.

Periodic points

Definition

Given a dynamical system $f : X \to X$ we say that a point $a \in X$ is periodic if $f^n(a) = a$ for some $n \in \mathbb{Z}_+$.

Question

Under what conditions on $f : X \to X$ are there "many" periodic points?

Periodic points for polynomials

Suppose that $f(x) = \sum_{i=0}^{N} a_i x^i \in \mathbb{C}[x]$ is a one-variable polynomial of degree N.

- So, when $N > 1$, $f^n(x) - x$ is a polynomial of degree N^n and it will have at least one solution (and one expects it to have about N^n solutions).
- When $N = 1$, whether or not $f^n(x) = x$ has solutions depends on whether or not a_1 is a root of unity and whether or not $a_0 = 0$. For example, $f(x) = x + 1$ has no periodic points.

Periodic points for polynomials

Suppose that $f(x) = \sum_{i=0}^{N} a_i x^i \in \mathbb{C}[x]$ is a one-variable polynomial of degree N.

A periodic point is a solution to the equation $f^n(x) = x$. How many such solutions do we expect?

- So, when $N > 1$, $f^n(x) - x$ is a polynomial of degree N^n and it will have at least one solution (and one expects it to have about N^n solutions).
- When $N = 1$, whether or not $f^n(x) = x$ has solutions depends on whether or not a_1 is a root of unity and whether or not $a_0 = 0$. For example, $f(x) = x + 1$ has no periodic points.

Periodic points for polynomials

Suppose that $f(x) = \sum_{i=0}^{N} a_i x^i \in \mathbb{C}[x]$ is a one-variable polynomial of degree N.

A periodic point is a solution to the equation $f^n(x) = x$. How many such solutions do we expect?

Composing f with itself many times, we compute

$$f^n(x) = a_N^{1+N+\cdots+N^{n-1}} x^{N^n} + \epsilon_n(x)$$

where $\deg(\epsilon_n(x)) < N^n$

- So, when $N > 1$, $f^n(x) - x$ is a polynomial of degree N^n and it will have at least one solution (and one expects it to have about N^n solutions).
- When $N = 1$, whether or not $f^n(x) = x$ has solutions depends on whether or not a_1 is a root of unity and whether or not $a_0 = 0$. For example, $f(x) = x + 1$ has no periodic points.

Periodic points for polynomials

Suppose that $f(x) = \sum_{i=0}^{N} a_i x^i \in \mathbb{C}[x]$ is a one-variable polynomial of degree N.

A periodic point is a solution to the equation $f^n(x) = x$. How many such solutions do we expect?

Composing f with itself many times, we compute

$$f^n(x) = a_N^{1+N+\cdots+N^{n-1}} x^{N^n} + \epsilon_n(x)$$

where $\deg(\epsilon_n(x)) < N^n$

- So, when $N > 1$, $f^n(x) - x$ is a polynomial of degree N^n and it will have at least one solution (and one expects it to have about N^n solutions).
- When $N = 1$, whether or not $f^n(x) = x$ has solutions depends on whether or not a_1 is a root of unity and whether or not $a_0 = 0$. For example, $f(x) = x + 1$ has no periodic points.

Periodic points for polynomials

Suppose that $f(x) = \sum_{i=0}^{N} a_i x^i \in \mathbb{C}[x]$ is a one-variable polynomial of degree N.

A periodic point is a solution to the equation $f^n(x) = x$. How many such solutions do we expect?

Composing f with itself many times, we compute

$$f^n(x) = a_N^{1+N+\cdots+N^{n-1}} x^{N^n} + \epsilon_n(x)$$

where $\deg(\epsilon_n(x)) < N^n$

- So, when $N > 1$, $f^n(x) - x$ is a polynomial of degree N^n and it will have at least one solution (and one expects it to have about N^n solutions).
- When $N = 1$, whether or not $f^n(x) = x$ has solutions depends on whether or not a_1 is a root of unity and whether or not $a_0 = 0$. For example, $f(x) = x + 1$ has no periodic points.

Fakhruddin's theorem

So, over $\mathbb{C}$ the question of whether or not a dynamical system has any periodic points could be delicate, but with $\mathbb{C}$ replaced by ${\mathbb{F}_p}^{\mathrm{alg}}$, the algebraic closure of the field of p-elements, there are always many periodic points.

Fakhruddin's theorem

So, over $\mathbb{C}$ the question of whether or not a dynamical system has any periodic points could be delicate, but with $\mathbb{C}$ replaced by ${\mathbb{F}_p}^{\mathrm{alg}}$, the algebraic closure of the field of p-elements, there are always many periodic points.

Theorem (Poonen (exposed by Fakhruddin))

Every algebraic dynamical system over ${\mathbb{F}_p}^{alg}$ has many periodic points unless there are obvious reasons why it does not.

Fakhruddin's theorem

So, over $\mathbb{C}$ the question of whether or not a dynamical system has any periodic points could be delicate, but with $\mathbb{C}$ replaced by $\mathbb{F}_p{}^{\mathrm{alg}}$, the algebraic closure of the field of p-elements, there are always many periodic points.

Theorem (Poonen (exposed by Fakhruddin))

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

We will return to a give a complete proof (modulo a theorem in logic) of this theorem and the highlighted terms from algebraic geometry will be defined precisely.

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

As it stands, $\sigma : K \to K$ is already a dynamical system and algebraic dynamical systems may be analyzed in terms of definable sets in difference fields.

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

As it stands, $\sigma : K \to K$ is already a dynamical system and algebraic dynamical systems may be analyzed in terms of definable sets in difference fields.

Example

- K any field, $\sigma = \mathrm{Id} : K \to K$ the identity function
- $K = \mathbb{C}(t)$ the field of rational functions in one variable over $\mathbb{C}$, $\sigma(f)(t) := f(t+1)$
- More generally, if $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$ and $K = \mathbb{C}(X)$ is the field of rational functions on X, then $\sigma : K \to K$ defined by $\sigma(g)(x) := g(f(x))$ makes K into a difference field.
- K a (perfect) field of characteristic $p > 0$, q a power of p, $\sigma = F_q : K \to K$, the q-power Frobenius: $x \mapsto x^q$

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

As it stands, $\sigma : K \to K$ is already a dynamical system and algebraic dynamical systems may be analyzed in terms of definable sets in difference fields.

Example

- K any field, $\sigma = \mathrm{Id} : K \to K$ the identity function
- $K = \mathbb{C}(t)$ the field of rational functions in one variable over $\mathbb{C}$, $\sigma(f)(t) := f(t+1)$
- More generally, if $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$ and $K = \mathbb{C}(X)$ is the field of rational functions on X, then $\sigma : K \to K$ defined by $\sigma(g)(x) := g(f(x))$ makes K into a difference field.
- K a (perfect) field of characteristic $p > 0$, q a power of p, $\sigma = F_q : K \to K$, the q-power Frobenius: $x \mapsto x^q$

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

As it stands, $\sigma : K \to K$ is already a dynamical system and algebraic dynamical systems may be analyzed in terms of definable sets in difference fields.

Example

- K any field, $\sigma = \mathrm{Id} : K \to K$ the identity function
- $K = \mathbb{C}(t)$ the field of rational functions in one variable over $\mathbb{C}$, $\sigma(f)(t) := f(t+1)$
- More generally, if $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$ and $K = \mathbb{C}(X)$ is the field of rational functions on X, then $\sigma : K \to K$ defined by $\sigma(g)(x) := g(f(x))$ makes K into a difference field.
- K a (perfect) field of characteristic $p > 0$, q a power of p, $\sigma = F_q : K \to K$, the q-power Frobenius: $x \mapsto x^q$

Difference fields

A difference field is a field K given together with a distinguished field endomorphism $\sigma : K \to K$.

As it stands, $\sigma : K \to K$ is already a dynamical system and algebraic dynamical systems may be analyzed in terms of definable sets in difference fields.

Example

- K any field, $\sigma = \mathrm{Id} : K \to K$ the identity function
- $K = \mathbb{C}(t)$ the field of rational functions in one variable over $\mathbb{C}$, $\sigma(f)(t) := f(t+1)$
- More generally, if $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$ and $K = \mathbb{C}(X)$ is the field of rational functions on X, then $\sigma : K \to K$ defined by $\sigma(g)(x) := g(f(x))$ makes K into a difference field.
- K a (perfect) field of characteristic $p > 0$, q a power of p, $\sigma = F_q : K \to K$, the q-power Frobenius: $x \mapsto x^q$

The theory of difference fields, expressed in the language $\mathscr{L}(+, \times, 0, 1, \sigma)$ has a model companion, ACFA.

- ACFA is a supersimple theory for which the rank one definable sets admit a fine structure theory along the lines of the Zilber trichotomy.
- On general grounds, we know that the axioms for ACFA say the structure is an existentially closed difference field and that the existentially closedness condition may be expressed in terms of certain resolvant formulas.

The theory of difference fields, expressed in the language $\mathscr{L}(+, \times, 0, 1, \sigma)$ has a model companion, ACFA.

- ACFA is a supersimple theory for which the rank one definable sets admit a fine structure theory along the lines of the Zilber trichotomy.
- On general grounds, we know that the axioms for ACFA say the structure is an existentially closed difference field and that the existentially closedness condition may be expressed in terms of certain resolvant formulas.

The theory of difference fields, expressed in the language $\mathscr{L}(+, \times, 0, 1, \sigma)$ has a model companion, ACFA.

- ACFA is a supersimple theory for which the rank one definable sets admit a fine structure theory along the lines of the Zilber trichotomy.
- On general grounds, we know that the axioms for ACFA say the structure is an existentially closed difference field and that the existentially closedness condition may be expressed in terms of certain resolvant formulas.

The theory of difference fields, expressed in the language $\mathscr{L}(+, \times, 0, 1, \sigma)$ has a model companion, ACFA.

- ACFA is a supersimple theory for which the rank one definable sets admit a fine structure theory along the lines of the Zilber trichotomy.
- On general grounds, we know that the axioms for ACFA say the structure is an existentially closed difference field and that the existentially closedness condition may be expressed in terms of certain resolvant formulas. Hrushovski presented much cleaner and more elegant geometric axioms.

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Geometric axioms for ACFA

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Geometric axioms for ACFA

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Geometric axioms for ACFA

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Definition

- A an algebraic variety X is irreducible if it cannot be expressed as the union of two proper subvarieties.
- A set $B \subseteq X$ is Zariski dense in X if for any proper subvariety $Y \subsetneq X$, the set $(B \smallsetminus Y)$ is nonempty.
- A regular map $f : X \to Y$ is dominant if the image of f is Zariski dense in Y.

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Definition

- A an algebraic variety X is irreducible if it cannot be expressed as the union of two proper subvarieties.
- A set $B \subseteq X$ is Zariski dense in X if for any proper subvariety $Y \subsetneq X$, the set $(B \smallsetminus Y)$ is nonempty.
- A regular map $f : X \to Y$ is dominant if the image of f is Zariski dense in Y.

Geometric axioms for ACFA

The class of existentially closed difference fields $(K, +, \times, 0, 1, \sigma)$ (ie the models of ACFA) is axiomatized by

- $(K, +, \times, 0, 1)$ is an algebraically closed field,
- $\sigma : K \to K$ is a field automorphism, and
- for each irreducible variety X over K and irreducible subvariety $Y \subseteq X \times X^\sigma$ for which the two projection maps are dominant, the set $\{a \in X : (a, \sigma(a)) \in Y\}$ is Zariski dense in X.

Definition

- A an algebraic variety X is irreducible if it cannot be expressed as the union of two proper subvarieties.
- A set $B \subseteq X$ is Zariski dense in X if for any proper subvariety $Y \subsetneq X$, the set $(B \smallsetminus Y)$ is nonempty.
- A regular map $f : X \to Y$ is dominant if the image of f is Zariski dense in Y.

Algebraic dynamics in ACFA

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \mathrm{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

Let L be a field extending K with an automorphism $\sigma : L \to L$ which restricts to the identity on K and make $(L, \sigma) \models \text{ACFA}$.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \text{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

Let L be a field extending K with an automorphism $\sigma : L \to L$ which restricts to the identity on K and make $(L, \sigma) \models \text{ACFA}$.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \text{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Algebraic dynamics in ACFA

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

Let L be a field extending K with an automorphism $\sigma : L \to L$ which restricts to the identity on K and make $(L, \sigma) \models \mathrm{ACFA}$.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \mathrm{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Algebraic dynamics in ACFA

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

Let L be a field extending K with an automorphism $\sigma : L \to L$ which restricts to the identity on K and make $(L, \sigma) \models \mathrm{ACFA}$.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \mathrm{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Hence, by the geometric axioms, the definable set $(X, f)^\sharp := \{x \in X(L) : f(x) = \sigma(x)\}$ is Zariski dense in X.

Algebraic dynamics in ACFA

Let $f : X \to X$ be an algebraic dynamical system defined over the field K with X irreducible and f dominant.

Let L be a field extending K with an automorphism $\sigma : L \to L$ which restricts to the identity on K and make $(L, \sigma) \models$ ACFA.

- $X^\sigma = X$ (since σ acts trivially on K)
- $Y := \mathrm{Graph}(f) = \{(x, y) \in X \times X : f(x) = y\}$ is an irreducible subvariety of $X \times X = X \times X^\sigma$ whose first projection map is onto X and whose second projection map is dominant as its image agrees with the image of f.

Hence, by the geometric axioms, the definable set $(X, f)^\sharp := \{x \in X(L) : f(x) = \sigma(x)\}$ is Zariski dense in X.

This is the starting point of the model theoretic approach to algebraic dynamics: understand algebraic geometric properties of the dynamical system $f : X \to X$ in terms of the model theoretic properties of the definable set $(X, f)^\sharp$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = {\mathbb{F}_p}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \smallsetminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \smallsetminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}} (\mathbb{F}_q^{alg}, F_q) \models \text{ACFA}$.
- Hence, by Łoś's theorem there is a q and $a \in X \smallsetminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \smallsetminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \smallsetminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}}(\mathbb{F}_q^{alg}, F_q) \models$ ACFA.
- Hence, by Łoś's theorem there is a q and $a \in X \smallsetminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \smallsetminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \smallsetminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}}(\mathbb{F}_q^{alg}, F_q) \models$ ACFA.
- Hence, by Łoś's theorem there is a q and $a \in X \smallsetminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \smallsetminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \smallsetminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}} (\mathbb{F}_q^{alg}, F_q) \models$ ACFA.
- Hence, by Łoś's theorem there is a q and $a \in X \smallsetminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \smallsetminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \smallsetminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}}(\mathbb{F}_q^{alg}, F_q) \models$ ACFA.
- Hence, by Łoś's theorem there is a q and $a \in X \smallsetminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Proof of Poonen's theorem

Theorem (Poonen)

If X is an irreducible variety defined over $K = \mathbb{F}_p{}^{alg}$ and $f : X \to X$ is a dominant self-map, then the set of periodic points for $f : X \to X$ is Zariski dense.

Proof.

- Let $Y \subsetneq X$ be a proper subvariety. We must find a periodic point $a \in X \setminus Y$.
- If we were in a model of ACFA in which X and Y were fixed, there would a point $a \in X \setminus Y$ with $\sigma(a) = f(a)$.
- Hrushovski showed that for every nonprincipal filter $\mathscr{U}$ on the set of powers of primes $\prod_{\mathscr{U}} (\mathbb{F}_q^{alg}, F_q) \models$ ACFA.
- Hence, by Łoś's theorem there is a q and $a \in X \setminus Y$ with $F_q(a) = \sigma(a) = f(a)$.
- As every element of $\mathbb{F}_q^{alg}$ is fixed by some power of the Frobenius, we can find some $\ell > 0$ for which $f^\ell(a) = F_q^\ell(a) = a$.

Definable sets in ACFA

Theorem (Chatzidakis, Hrushovski, Peterzil)

ACFA satisfies the Zilber trichotomy.

- X is trivial in the sense that all induced dependencies on X are essentially binary,
- X is group-like in the sense that there is a finite-to-finite definable correspondence with a definable abelian group G all of whose (quantifier free) definable sets in all Cartesian powers are finite Boolean combinations of cosets, or
- X is field-like in the sense that there is a definable field F and a definable correspondence between X and F.

Definable sets in ACFA

Theorem (Chatzidakis, Hrushovski, Peterzil)

ACFA satisfies the Zilber trichotomy.

That is, if X is a (type) definable set in a model of ACFA of dimension (Lascar rank) one, then either

- X is trivial in the sense that all induced dependencies on X are essentially binary,
- X is group-like in the sense that there is a finite-to-finite definable correspondence with a definable abelian group G all of whose (quantifier free) definable sets in all Cartesian powers are finite Boolean combinations of cosets, or
- X is field-like in the sense that there is a definable field F and a definable correspondence between X and F.

Theorem (Chatzidakis, Hrushovski, Peterzil)

ACFA satisfies the Zilber trichotomy.

That is, if X is a (type) definable set in a model of ACFA of dimension (Lascar rank) one, then either

- X is trivial in the sense that all induced dependencies on X are essentially binary,
- X is group-like in the sense that there is a finite-to-finite definable correspondence with a definable abelian group G all of whose (quantifier free) definable sets in all Cartesian powers are finite Boolean combinations of cosets, or
- X is field-like in the sense that there is a definable field F and a definable correspondence between X and F.

Definable sets in ACFA

Theorem (Chatzidakis, Hrushovski, Peterzil)

ACFA satisfies the Zilber trichotomy.

That is, if X is a (type) definable set in a model of ACFA of dimension (Lascar rank) one, then either

- X is trivial in the sense that all induced dependencies on X are essentially binary,
- X is group-like in the sense that there is a finite-to-finite definable correspondence with a definable abelian group G all of whose (quantifier free) definable sets in all Cartesian powers are finite Boolean combinations of cosets, or
- X is field-like in the sense that there is a definable field F and a definable correspondence between X and F.

Definable sets in ACFA

Theorem (Chatzidakis, Hrushovski, Peterzil)

ACFA satisfies the Zilber trichotomy.

That is, if X is a (type) definable set in a model of ACFA of dimension (Lascar rank) one, then either

- X is trivial in the sense that all induced dependencies on X are essentially binary,
- X is group-like in the sense that there is a finite-to-finite definable correspondence with a definable abelian group G all of whose (quantifier free) definable sets in all Cartesian powers are finite Boolean combinations of cosets, or
- X is field-like in the sense that there is a definable field F and a definable correspondence between X and F.

If $f : X \to X$ is defined over the fixed field of a model (K, σ) of ACFA, then the set $(X, f)^\sharp = \{x \in X : f(x) = \sigma(x)\}$ has finite Lascar rank and may be analyzed in terms of types of Lascar rank one.

Some consequences for dynamics

If $f : X \to X$ is defined over the fixed field of a model (K, σ) of ACFA, then the set $(X, f)^\sharp = \{x \in X : f(x) = \sigma(x)\}$ has finite Lascar rank and may be analyzed in terms of types of Lascar rank one.

Definition

If $f : X \to X$ is a dynamical system, then an invariant subvariety is a subvariety $Y \subseteq X$ for which $f(Y) \subseteq Y$. A periodic subvariety is a subvariety $X \subseteq X$ which is invariant under f^n for some $n \in \mathbb{Z}_+$.

If $f : X \to X$ is defined over the fixed field of a model (K, σ) of ACFA, then the set $(X, f)^\sharp = \{x \in X : f(x) = \sigma(x)\}$ has finite Lascar rank and may be analyzed in terms of types of Lascar rank one.

Definition

If $f : X \to X$ is a dynamical system, then an invariant subvariety is a subvariety $Y \subseteq X$ for which $f(Y) \subseteq Y$. A periodic subvariety is a subvariety $X \subseteq X$ which is invariant under f^n for some $n \in \mathbb{Z}_+$.

If $Y \subseteq X$ is a periodic subvariety (of period n, say) also defined over the fixed field, then it corresponds to the definable subset $\{x \in Y : \sigma^n(x) = f^n(x)\}$ of $(X, f)^\sharp$, which is Zariski dense in $\overline{f(Y)}$.

If $f : X \to X$ is defined over the fixed field of a model (K, σ) of ACFA, then the set $(X, f)^\sharp = \{x \in X : f(x) = \sigma(x)\}$ has finite Lascar rank and may be analyzed in terms of types of Lascar rank one.

Definition

If $f : X \to X$ is a dynamical system, then an invariant subvariety is a subvariety $Y \subseteq X$ for which $f(Y) \subseteq Y$. A periodic subvariety is a subvariety $X \subseteq X$ which is invariant under f^n for some $n \in \mathbb{Z}_+$.

If $Y \subseteq X$ is a periodic subvariety (of period n, say) also defined over the fixed field, then it corresponds to the definable subset $\{x \in Y : \sigma^n(x) = f^n(x)\}$ of $(X, f)^\sharp$, which is Zariski dense in $\overline{f(Y)}$.

Corollary

Unless $(X, f)^\sharp$ is analyzable by field-like sets, periodic subvarieties of $f : X \to X$ are "rare."

A more precise version?

- How can one tell into which class $(X, f)^\sharp$ belongs?
- Can one deduce more precise information about the induced structure than simply the coarse trichotomy?

A more precise version?

- How can one tell into which class $(X, f)^\sharp$ belongs?
- Can one deduce more precise information about the induced structure than simply the coarse trichotomy?

Answers in one variable

For now let $(K, \sigma) \models \mathrm{ACFA}$.

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

Proposition

The following are equivalent.

- $(K, f)^\sharp$ *is field-like.*
- $f : K \to K$ *is a bijection.*
- $f(x) = \frac{Ax^{p^m}+B}{Cx^{p^m}+D}$ *for some invertible matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is 1 if the characteristic of* K *is zero).*
- *There is a definable bijection* $h : (K, f)^\sharp \to (K, x \mapsto x^{p^m})$ *for some* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is 1 if the characteristic of* K *is zero).*

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

Proposition

The following are equivalent.

- $(K, f)^\sharp$ *is field-like.*
- $f : K \to K$ *is a bijection.*
- $f(x) = \frac{Ax^{p^m}+B}{Cx^{p^m}+D}$ *for some invertible matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*
- *There is a definable bijection* $h : (K, f)^\sharp \to (K, x \mapsto x^{p^m})$ *for some* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

Proposition

The following are equivalent.

- $(K, f)^\sharp$ *is field-like.*
- $f : K \to K$ *is a bijection.*
- $f(x) = \frac{Ax^{p^m}+B}{Cx^{p^m}+D}$ *for some invertible matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*
- *There is a definable bijection* $h : (K, f)^\sharp \to (K, x \mapsto x^{p^m})$ *for some* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

Proposition

The following are equivalent.

- $(K, f)^\sharp$ *is field-like.*
- $f : K \to K$ *is a bijection.*
- $f(x) = \frac{Ax^{p^m}+B}{Cx^{p^m}+D}$ *for some invertible matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*
- *There is a definable bijection* $h : (K, f)^\sharp \to (K, x \mapsto x^{p^m})$ *for some* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*

For now let $(K, \sigma) \models \mathrm{ACFA}$.

If $f : K \to K$ is a nonconstant rational function, then the set $(K, f)^\sharp := \{x \in K : \sigma(x) = f(x)\}$ has Lascar rank one.

Proposition

The following are equivalent.

- $(K, f)^\sharp$ *is field-like.*
- $f : K \to K$ *is a bijection.*
- $f(x) = \frac{Ax^{p^m}+B}{Cx^{p^m}+D}$ *for some invertible matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*
- *There is a definable bijection* $h : (K, f)^\sharp \to (K, x \mapsto x^{p^m})$ *for some* $m \in \mathbb{N}$ *where* p *is the characteristic of* K *(or is* 1 *if the characteristic of* K *is zero).*

Groups in one variable

Suppose that $f(x) = x^N$ for some $N \in \mathbb{Z}_+$.

- $(K^\times, f)^\sharp = \{x \in K : x \neq 0 \text{ and } f(x) = \sigma(x)\}$ is a group under multiplication.
- As long as $N > 1$ and is not a power of the characteristic, $f(x)$ is not of the form $\frac{Ax^{p^m}+B}{Cx^{p^m}+D}$.
- Hence, under these conditions, by the trichotomy theorem, $(K^\times, f)$ must be "group-like" in the sense that every definable subset of any Cartesian power is a finite Boolean combination of definable subgroups.

Groups in one variable

Suppose that $f(x) = x^N$ for some $N \in \mathbb{Z}_+$.

- $(K^\times, f)^\sharp = \{x \in K : x \neq 0 \text{ and } f(x) = \sigma(x)\}$ is a group under multiplication.
- As long as $N > 1$ and is not a power of the characteristic, $f(x)$ is not of the form $\frac{Ax^{p^m}+B}{Cx^{p^m}+D}$.
- Hence, under these conditions, by the trichotomy theorem, $(K^\times, f)$ must be "group-like" in the sense that every definable subset of any Cartesian power is a finite Boolean combination of definable subgroups.

Groups in one variable

Suppose that $f(x) = x^N$ for some $N \in \mathbb{Z}_+$.

- $(K^\times, f)^\sharp = \{x \in K : x \neq 0 \text{ and } f(x) = \sigma(x)\}$ is a group under multiplication.
- As long as $N > 1$ and is not a power of the characteristic, $f(x)$ is not of the form $\frac{Ax^{p^m}+B}{Cx^{p^m}+D}$.
- Hence, under these conditions, by the trichotomy theorem, $(K^\times, f)$ must be "group-like" in the sense that every definable subset of any Cartesian power is a finite Boolean combination of definable subgroups.

Groups in one variable

Suppose that $f(x) = x^N$ for some $N \in \mathbb{Z}_+$.

- $(K^\times, f)^\sharp = \{x \in K : x \neq 0 \text{ and } f(x) = \sigma(x)\}$ is a group under multiplication.
- As long as $N > 1$ and is not a power of the characteristic, $f(x)$ is not of the form $\frac{Ax^{p^m}+B}{Cx^{p^m}+D}$.
- Hence, under these conditions, by the trichotomy theorem, $(K^\times, f)$ must be "group-like" in the sense that every definable subset of any Cartesian power is a finite Boolean combination of definable subgroups.

Suppose that $f(x) = x^N$ for some $N \in \mathbb{Z}_+$.

- $(K^\times, f)^\sharp = \{x \in K : x \neq 0 \text{ and } f(x) = \sigma(x)\}$ is a group under multiplication.
- As long as $N > 1$ and is not a power of the characteristic, $f(x)$ is not of the form $\frac{Ax^{p^m}+B}{Cx^{p^m}+D}$.
- Hence, under these conditions, by the trichotomy theorem, $(K^\times, f)$ must be "group-like" in the sense that every definable subset of any Cartesian power is a finite Boolean combination of definable subgroups.

Question

How do the other group-like sets of the form $(K, f)^\sharp$ look?

Other group-like rational functions

- If K has characteristic $p > 0$, then the set $\{x \in K : \sigma(x) = \sum a_i x^{p^i}\}$ is a group-like subgroup of $(K, +)$ provided that at least two of the coefficients are nonzero.
- $f(x) = C_2(x) = x^2 - 2$, the second Chebyshev polynomial, satisfies $C_2(x + \frac{1}{x}) = x^2 + \frac{1}{x^2}$.
- More generally, if G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^\sigma \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^\sigma} \\ K & \xrightarrow{f} & K \end{array}$$

commutes, then $(K, f)^\sharp$ is group-like.

Other group-like rational functions

- If K has characteristic $p > 0$, then the set $\{x \in K : \sigma(x) = \sum a_i x^{p^i}\}$ is a group-like subgroup of $(K, +)$ provided that at least two of the coefficients are nonzero.
- $f(x) = C_2(x) = x^2 - 2$, the second Chebyshev polynomial, satisfies $C_2(x + \frac{1}{x}) = x^2 + \frac{1}{x^2}$.
- More generally, if G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & G^\sigma \arrow[d, "\pi^\sigma"] \\ K \arrow[r, "f"] & K \end{tikzcd}$$

commutes, then $(K, f)^\sharp$ is group-like.

Other group-like rational functions

- If K has characteristic $p > 0$, then the set $\{x \in K : \sigma(x) = \sum a_i x^{p^i}\}$ is a group-like subgroup of $(K, +)$ provided that at least two of the coefficients are nonzero.
- $f(x) = C_2(x) = x^2 - 2$, the second Chebyshev polynomial, satisfies $C_2(x + \frac{1}{x}) = x^2 + \frac{1}{x^2}$. Hence, the rational function $g : K \to K$ given by $x \mapsto x + \frac{1}{x}$ takes $(K, x \mapsto x^2)^\sharp$ to $(K, C_2)^\sharp$. Thus, $(K, C_2)^\sharp$ is group-like.
- More generally, if G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & G^\sigma \arrow[d, "\pi^\sigma"] \\ K \arrow[r, "f"] & K \end{tikzcd}$$

commutes, then $(K, f)^\sharp$ is group-like.

Other group-like rational functions

- If K has characteristic $p > 0$, then the set $\{x \in K : \sigma(x) = \sum a_i x^{p^i}\}$ is a group-like subgroup of $(K, +)$ provided that at least two of the coefficients are nonzero.
- $f(x) = C_2(x) = x^2 - 2$, the second Chebyshev polynomial, satisfies $C_2(x + \frac{1}{x}) = x^2 + \frac{1}{x^2}$. Hence, the rational function $g : K \to K$ given by $x \mapsto x + \frac{1}{x}$ takes $(K, x \mapsto x^2)^\sharp$ to $(K, C_2)^\sharp$. Thus, $(K, C_2)^\sharp$ is group-like.
- More generally, if G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^\sigma \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^\sigma} \\ K & \xrightarrow{f} & K \end{array}$$

commutes, then $(K, f)^\sharp$ is group-like.

Classification of group-like rational functions

If G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^\sigma \\ \pi \downarrow & & \downarrow \pi^\sigma \\ K & \xrightarrow{f} & K \end{array}$$

commutes, then $(K, f)^\sharp$ is group-like.

Classification of group-like rational functions

If G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^\sigma \\ \pi \downarrow & & \downarrow \pi^\sigma \\ K & \xrightarrow{f} & K \end{array}$$

commutes, then $(K, f)^\sharp$ is group-like.

Theorem (Medvedev)

This is the only way a rational function $f : K \to K$ may produce a group-like $(K, f)^\sharp$. Moreover, given a rational function f one may bound the degree in the search for the relevant π as a function of the degree of f.

Classification of group-like rational functions

If G is a connected algebraic group of dimension one, $\phi : G \to G^\sigma$ is a map of algebraic groups with a nontrivial kernel, and $\pi : G \to K$ is a rational function for which

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^\sigma \\ \pi \downarrow & & \downarrow \pi^\sigma \\ K & \xrightarrow{f} & K \end{array}$$

commutes, then $(K, f)^\sharp$ is group-like.

Theorem (Medvedev)

This is the only way a rational function $f : K \to K$ may produce a group-like $(K, f)^\sharp$. Moreover, given a rational function f one may bound the degree in the search for the relevant π as a function of the degree of f.

It follows that for any reasonable sense of "most," most sets of the form $(K, f)^\sharp$ are trivial.

Trivial sets

If $f : K \to K$ is a nonconstant rational function not covered by a group, then $(K, f)^\sharp$ is trivial.

Trivial sets

If $f : K \longrightarrow K$ is a nonconstant rational function not covered by a group, then $(K, f)^\sharp$ is trivial. Hence, if $X \subseteq K^n$ is irreducible and (skew)-periodic for f, X must be a component of a variety defined by finitely many equations of the form $G(x_i, x_j) = 0$.

Trivial sets

If $f : K \to K$ is a nonconstant rational function not covered by a group, then $(K, f)^\sharp$ is trivial. Hence, if $X \subseteq K^n$ is irreducible and (skew)-periodic for f, X must be a component of a variety defined by finitely many equations of the form $G(x_i, x_j) = 0$.

Question

What can we say about the equations $G(x_i, x_j) = 0$?

If $f : K \to K$ is a nonconstant rational function not covered by a group, then $(K, f)^\sharp$ is trivial. Hence, if $X \subseteq K^n$ is irreducible and (skew)-periodic for f, X must be a component of a variety defined by finitely many equations of the form $G(x_i, x_j) = 0$.

Question

What can we say about the equations $G(x_i, x_j) = 0$?

In general, the situation is still opaque, but if we restrict to characteristic zero and f being a polynomial, we can completely classify the possible equations.

Trivial sets

If $f : K \to K$ is a nonconstant rational function not covered by a group, then $(K, f)^\sharp$ is trivial. Hence, if $X \subseteq K^n$ is irreducible and (skew)-periodic for f, X must be a component of a variety defined by finitely many equations of the form $G(x_i, x_j) = 0$.

Question

What can we say about the equations $G(x_i, x_j) = 0$?

In general, the situation is still opaque, but if we restrict to characteristic zero and f being a polynomial, we can completely classify the possible equations.

Theorem (Medvedev, Scanlon)

Suppose that K has characteristic zero and f is a polynomial with coefficients from an algebraically closed subfield of the fixed field of σ and that f cannot be expressed as a nontrivial compositional power. If $(K, f)^\sharp$ is trivial, then every irreducible f-periodic subvariety of K^n is a component of a variety defined by finitely many equations of the form $f^\ell(x_i) = f^k(x_j)$.

A number theoretic implication

Theorem

Let p be a prime number and $g(x) \in \mathbb{Z}[x]$ a polynomial with integer coefficients of degree at most p. Set $f(x) = x^p + pg(x)$. We assume that f is not linearly conjugate to x^p or the p^{th} Chebyshev polynomial. Then if $X \subseteq \mathbb{C}^n$ is an irreducible variety containing a Zariski dense set of points of the form $(\zeta_1, \ldots, \zeta_n)$ where each ζ_i is f-periodic, X must be a component of a variety defined by equations of the form $f^\ell(x_i) = f^k(x_j)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\text{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\text{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms. Set $h(0) := \infty$.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\text{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights

As a rule of thumb, if $f : X \to X$ is an algebraic dynamical system and $a \in X$, then $f(a)$ "more complicated" than a unless a is a preperiodic point.

Here, "more complicated" means "of larger height." Let me explain with an example.

On $\mathbb{Q}$ one may define the logarithmic height, $h : \mathbb{Q}^\times \to \mathbb{R}_{\geq 0}$, by $h(\frac{a}{b}) := \max\{\ln|a|, \ln|b|\}$ as long as $\frac{a}{b}$ is written in lowest terms.

- For any $d \in \mathbb{Z}_+$ we have $h(x^d) = dh(x)$.
- More generally, if $f(x)$ is a polynomial of degree $d \geq 2$, then $h(f(x)) \approx dh(x)$.
- Defining the canonical height, $\widehat{h}(x) := \lim h(f^n(x))/d^n$, we obtain $\widehat{h}(f(x)) = d\widehat{h}(x)$.
- Even more generally, if $f : X \to X$ is a "polarized" algebraic dynamical system, there is an associated integer $d \geq 2$ and a canonical height function $\widehat{h} : X(\mathbb{Q}^{\text{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $\widehat{h}(f(x)) = d\widehat{f}(x)$.

Heights on function fields

On $\mathbb{C}(t)$, define $h(f/g) := \max\{\deg(f), \deg(g)\}$, again when f and g are relatively prime polynomials.

On $\mathbb{C}(t)$, define $h(f/g) := \max\{\deg(f), \deg(g)\}$, again when f and g are relatively prime polynomials.

Note $h(f) = 0$ iff $f \in \mathbb{C}^\times$.

Heights on function fields

On $\mathbb{C}(t)$, define $h(f/g) := \max\{\deg(f), \deg(g)\}$, again when f and g are relatively prime polynomials.

Note $h(f) = 0$ iff $f \in \mathbb{C}^\times$.

As with dynamical systems over $\mathbb{Q}$, if $f : X \to X$ is a polarized dynamical system over $\mathbb{C}(t)$, (it would not hurt to think of the case that f is given by a rational function of degree at least two) then there is a canonical height $\widehat{h} : X(\mathbb{C}(t)^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $|\widehat{h}(x) - h(x)|$ is bounded and $\widehat{h} \circ f = d \cdot \widehat{h}$ for some $d \geq 2$.

On $\mathbb{C}(t)$, define $h(f/g) := \max\{\deg(f), \deg(g)\}$, again when f and g are relatively prime polynomials.

Note $h(f) = 0$ iff $f \in \mathbb{C}^\times$.

As with dynamical systems over $\mathbb{Q}$, if $f : X \to X$ is a polarized dynamical system over $\mathbb{C}(t)$, (it would not hurt to think of the case that f is given by a rational function of degree at least two) then there is a canonical height $\widehat{h} : X(\mathbb{C}(t)^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $|\widehat{h}(x) - h(x)|$ is bounded and $\widehat{h} \circ f = d \cdot \widehat{h}$ for some $d \geq 2$.

If $f : X \to X$ were actually defined over $\mathbb{C}$, then for any $a \in X(\mathbb{C})$, we would have $f(a) \in X(\mathbb{C})$ also.

On $\mathbb{C}(t)$, define $h(f/g) := \max\{\deg(f), \deg(g)\}$, again when f and g are relatively prime polynomials.

Note $h(f) = 0$ iff $f \in \mathbb{C}^\times$.

As with dynamical systems over $\mathbb{Q}$, if $f : X \to X$ is a polarized dynamical system over $\mathbb{C}(t)$, (it would not hurt to think of the case that f is given by a rational function of degree at least two) then there is a canonical height $\widehat{h} : X(\mathbb{C}(t)^{\mathrm{alg}}) \to \mathbb{R}_{\geq 0}$ satisfying $|\widehat{h}(x) - h(x)|$ is bounded and $\widehat{h} \circ f = d \cdot \widehat{h}$ for some $d \geq 2$.

If $f : X \to X$ were actually defined over $\mathbb{C}$, then for any $a \in X(\mathbb{C})$, we would have $f(a) \in X(\mathbb{C})$ also. Hence, for every n we would have $h(f^n(a)) = 0$ so that $\widehat{h}(a) = \lim h(f^n(a))/d^n = 0$.

Theorem (M. Baker)

Suppose that $f \in \mathbb{C}(t,s)$ is a rational function in the variables t and s over $\mathbb{C}$ which when considered as a rational function in the variable s with coefficients from $\mathbb{C}(t)$ has degree at least two. If there is some non-f-preperiodic point $a \in \mathbb{C}(t)$ with $\widehat{h}(a) = 0$, then f is essentially defined over $\mathbb{C}$ in the sense that there is a rational function $\widetilde{f} \in \mathbb{C}(s)$ and a fractional linear transformation γ defined over $\mathbb{C}(t)$ with $f = \gamma^{-1} \circ \widetilde{f} \circ \gamma$.

Theorem (M. Baker)

Suppose that $f \in \mathbb{C}(t,s)$ is a rational function in the variables t and s over $\mathbb{C}$ which when considered as a rational function in the variable s with coefficients from $\mathbb{C}(t)$ has degree at least two. If there is some non-f-preperiodic point $a \in \mathbb{C}(t)$ with $\widehat{h}(a) = 0$, then f is essentially defined over $\mathbb{C}$ in the sense that there is a rational function $\widetilde{f} \in \mathbb{C}(s)$ and a fractional linear transformation γ defined over $\mathbb{C}(t)$ with $f = \gamma^{-1} \circ \widetilde{f} \circ \gamma$.

Baker's proof is geometric and uses capacity theory.

Logical reformulation: bounded sets

$\mathbb{C}(t)$ is not interpretable in $\mathbb{C}$, but it is naturally expressed as a countable union of sets interpretable in $\mathbb{C}$ on which the algebraic operations of $\mathbb{C}(t)$ are piecewise definable.

Logical reformulation: bounded sets

$\mathbb{C}(t)$ is not interpretable in $\mathbb{C}$, but it is naturally expressed as a countable union of sets interpretable in $\mathbb{C}$ on which the algebraic operations of $\mathbb{C}(t)$ are piecewise definable.

As such, relative to this identification, every quantifier-free definable set in $\mathbb{C}(t)^m$ may be understood as a countable union of sets interpretable in $\mathbb{C}$.

Logical reformulation: bounded sets

$\mathbb{C}(t)$ is not interpretable in $\mathbb{C}$, but it is naturally expressed as a countable union of sets interpretable in $\mathbb{C}$ on which the algebraic operations of $\mathbb{C}(t)$ are piecewise definable.

As such, relative to this identification, every quantifier-free definable set in $\mathbb{C}(t)^m$ may be understood as a countable union of sets interpretable in $\mathbb{C}$.

Definition

Let X be a variety defined over $\mathbb{C}(t)$. We say that $B \subseteq X(\mathbb{C}(t))$ is bounded if relative to the natural presentation of $\mathbb{C}(t)$ as a countable union of sets defined in $\mathbb{C}$, B is contained in a union of *finitely* many sets defined in $\mathbb{C}$.

Logical reformulation: a theorem

Theorem (Chatzidakis, Hrushovski)

Let $f : X \to X$ be a dominant algebraic dynamical system defined over $\mathbb{C}(t)$ assumed to be primitive. Then there is an infinite bounded f-invariant subset of $X(\mathbb{C}(t))$ if and only if $f : X \to X$ is isomorphic to a dynamical system defined over $\mathbb{C}$.

- The theorem on heights follows, as the set $\{a \in X(\mathbb{C}(t)) : \widehat{h}(a) = 0\}$ is necessarily bounded, but would contain the infinite orbit $\mathscr{O}_f(a) := \{f^n(a) : n \in \mathbb{N}\}$ if there were some non-preperiodic point a.
- The proof proceeds by extracting from the invariant bounded set a dynamical system $g : Y \to Y$ defined over $\mathbb{C}$ which is in some obvious way *related* to $f : X \to X$.
- The isomorphism is produced via the theory of quantifier-free canonical bases.

Logical reformulation: a theorem

Theorem (Chatzidakis, Hrushovski)

Let $f : X \to X$ be a dominant algebraic dynamical system defined over $\mathbb{C}(t)$ assumed to be primitive. Then there is an infinite bounded f-invariant subset of $X(\mathbb{C}(t))$ if and only if $f : X \to X$ is isomorphic to a dynamical system defined over $\mathbb{C}$.

- The theorem on heights follows, as the set $\{a \in X(\mathbb{C}(t)) : \widehat{h}(a) = 0\}$ is necessarily bounded, but would contain the infinite orbit $\mathscr{O}_f(a) := \{f^n(a) : n \in \mathbb{N}\}$ if there were some non-preperiodic point a.
- The proof proceeds by extracting from the invariant bounded set a dynamical system $g : Y \to Y$ defined over $\mathbb{C}$ which is in some obvious way *related* to $f : X \to X$.
- The isomorphism is produced via the theory of quantifier-free canonical bases.

Logical reformulation: a theorem

Theorem (Chatzidakis, Hrushovski)

Let $f : X \to X$ be a dominant algebraic dynamical system defined over $\mathbb{C}(t)$ assumed to be primitive. Then there is an infinite bounded f-invariant subset of $X(\mathbb{C}(t))$ if and only if $f : X \to X$ is isomorphic to a dynamical system defined over $\mathbb{C}$.

- The theorem on heights follows, as the set $\{a \in X(\mathbb{C}(t)) : \widehat{h}(a) = 0\}$ is necessarily bounded, but would contain the infinite orbit $\mathscr{O}_f(a) := \{f^n(a) : n \in \mathbb{N}\}$ if there were some non-preperiodic point a.
- The proof proceeds by extracting from the invariant bounded set a dynamical system $g : Y \to Y$ defined over $\mathbb{C}$ which is in some obvious way *related* to $f : X \to X$.
- The isomorphism is produced via the theory of quantifier-free canonical bases.

Logical reformulation: a theorem

Theorem (Chatzidakis, Hrushovski)

Let $f : X \to X$ be a dominant algebraic dynamical system defined over $\mathbb{C}(t)$ assumed to be primitive. Then there is an infinite bounded f-invariant subset of $X(\mathbb{C}(t))$ if and only if $f : X \to X$ is isomorphic to a dynamical system defined over $\mathbb{C}$.

- The theorem on heights follows, as the set $\{a \in X(\mathbb{C}(t)) : \widehat{h}(a) = 0\}$ is necessarily bounded, but would contain the infinite orbit $\mathscr{O}_f(a) := \{f^n(a) : n \in \mathbb{N}\}$ if there were some non-preperiodic point a.
- The proof proceeds by extracting from the invariant bounded set a dynamical system $g : Y \to Y$ defined over $\mathbb{C}$ which is in some obvious way *related* to $f : X \to X$.
- The isomorphism is produced via the theory of quantifier-free canonical bases.

References

- M. Baker, A Finiteness Theorem for Canonical Heights Attached to Rational Maps over Function Fields, *J. Reine Angew. Math.* (to appear)
- Z. Chatzidakis and E. Hrushovski, Difference fields and descent in algebraic dynamics I and II, (to appear)
- N. Fakhruddin, Questions on self maps of algebraic varieties, *J. Ramanujan Math. Soc.* **18** (2003), no. 2, 109–122.
- A. Medvedev, Minimal sets in ACFA, PhD thesis, Univ. of California, Berkeley, May 2007.
- A. Medvedev and T. Scanlon, Polynomial dynamics, in preparation.