Projective absoluteness and thin equivalence relations

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Outline

1. Motivation
   - Definitions and previous results
2. Results
   - Thin equivalence relations and reasonable forcing
   - Thin equivalence relations and Σ^1_2 c.c.c. forcing

Projective absoluteness

Definition

Generic Σ^1_n absoluteness holds for a forcing $\mathbb{P}$ if $V \prec_{\Sigma^1_n} V^{\mathbb{P}}$.

Theorem

(Shoenfield 1961) Σ^1_2 absoluteness holds for any forcing.

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(Woodin 1981) $\underset{\sim}{\Pi}^1_n$ determinacy implies Σ^1_{n+2} absoluteness for Cohen forcing and random forcing.

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Thin equivalence relations

Definition

An equivalence relation E on Baire space (the reals) is thin if there is no perfect set (i.e. a nonempty closed set without isolated points) of pairwise inequivalent reals.

A prewellorder is a wellfounded linear preorder on the reals.

Fact

Every prewellorder induces an equivalence relation.

If $\underset{\sim}{\Pi}^1_n$ determinacy holds, then all $\underset{\sim}{\Sigma}^1_{n+1}$ and $\underset{\sim}{\Pi}^1_{n+1}$ prewellorders induce thin equivalence relations.

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Equivalence relations in forcing extensions

Definition

If a set of reals E is given by a fixed definition, we write E for the set in $V^{\mathbb{P}}$ with the same definition.

Fact

If E is a thin provably $\underset{\sim}{\Delta}^1_{n+1}$ equivalence relation and generic Σ^1_n absoluteness holds for $\mathbb{P}$, then $E^{V^{\mathbb{P}}} \cap V = E$.

The issue is whether the forcing introduces new equivalence classes to E.

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Equivalence relations and absoluteness

Lemma

Suppose a forcing $\mathbb{P}$ does not introduce new equivalence classes to all thin provably $\underset{\sim}{\Delta}^1_n$ equivalence relations. Then Σ^1_{n+1} absoluteness holds for $\mathbb{P}$.

Proof:
We want to show Σ^1_{k+1} absoluteness by induction for $k \leq n$. Suppose ϕ is a Π^1_k formula and there is some real $r \in V^{\mathbb{P}}$ with $V^{\mathbb{P}} \models \phi(r)$. Define $(x, y) \in E$ iff

$$(\phi(x) \wedge \phi(y)) \vee (\neg\phi(x) \wedge \neg\phi(y))$$

Then there is a real x in V with $\phi(x)$.

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The effect of large cardinals

Theorem

(Foreman, Magidor 1995) Let E be a thin equivalence relation with a tree representation which is absolute for forcing of size $< \kappa$. Then reasonable forcing of size $< \kappa$ does not introduce new equivalence classes to E.

Combined with a result of Martin and Steel this shows

Theorem

Suppose κ is a limit of Woodin cardinals. Then reasonable forcing of size $< \kappa$ does not add equivalence classes to thin projective equivalence relations.

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$M_n^\#$

$M_n^\#$ is a transitive set of the form $J_\alpha[\vec{E}]$ where $\vec{E}$ is a sequence of extenders. It is iterable in the sense of iteration trees and contains n Woodin cardinals. More exactly:

Definition

$M_n^\#(X)$ is a minimal ω_1-iterable sound X-premouse with $\rho_1 \leq \sup(tc(X \cup \omega))$ and an extender above n Woodin cardinals, if this exists.

Theorem

(Harrington, Martin, Neeman, Woodin) The existence of $M_n^\#(x)$ for all reals x is equivalent to $\underset{\sim}{\Pi}^1_{n+1}$ determinacy.

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Properties of $M_n^\#$

Lemma

(Woodin) $M_n^\#(X)$ can calculate which Σ^1_{n+2} statements are true in V.
For even n we have $M_n^\#(X) \prec_{\Sigma^1_{n+2}} V$.
For odd n any Σ^1_{n+2} statement holds in V iff it is forced over $M_n^\#(X)$ by collapsing the least Woodin cardinal.

Hence if $M_n^\#(x)$ is absolute for every real x we get $V \prec_{\Sigma^1_{n+2}} V^{\mathbb{P}}$.

Lemma

(Folklore) Suppose $M_n^\#(X)$ exists for every $X \in H_{\kappa^+}$ and is κ^+-iterable. Then this also holds in $V^{\mathbb{P}}$ for every forcing $\mathbb{P}$ of size κ.

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Projective absoluteness

Lemma

(S.) Suppose $M_n^\#(X)$ exists for every $X \in H_{\kappa^+}$ and is κ^+-iterable. Then Σ^1_{n+3}-absoluteness holds for every reasonable forcing $\mathbb{P}$ of size κ.

The proof is a simple case of the proof of the next theorem.

Key lemma

If $\mathbb{P}$ is a forcing and τ is a $\mathbb{P}$-name, we write τ and τ' for the corresponding $\mathbb{P} \times \mathbb{P}$-names.

Lemma

(S.) Let E be a thin $\underset{\sim}{\Pi}^1_{n+3}$ equivalence relation. Suppose $\mathbb{P}$ is a forcing of size κ and $M_n^{\#}(X)$ exists for every $X \in H_{\kappa^+}$. Then for every $\mathbb{P}$-name τ for a real the set

$$D := \{p \in \mathbb{P} : (p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau E \tau'\}$$

is dense.

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Proof

- suppose E is $\Pi^1_{n+3}(z)$
- suppose D is not dense
- there is a condition $p \in \mathbb{P}$ so that for every $q \leq p$ there are $r, u \leq q$ with

$$(r, s) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg \tau E \tau'$$

- let $H \prec V_\lambda$ be countable with $\mathbb{P}, p, z, \tau \in H$, where V_λ is sufficiently elementary in V
- let $\pi : \bar{H} \to H$ be the uncollapsing map, $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{p}) = p$, and $\pi(\bar{\tau}) = \tau$

Proof

We enumerate the open dense subsets in $\bar{H}$ of $\bar{\mathbb{P}} \times \bar{\mathbb{P}}$ as $(D_n : n < \omega)$ and construct a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ such that

- $p_\emptyset = \bar{p}$
- $p_s \leq p_t$ if $t \subseteq s$
- $(p_{s^\frown 0}, p_{s^\frown 1}) \Vdash_{\bar{\mathbb{P}} \times \bar{\mathbb{P}}} \neg \bar{\tau} E \bar{\tau}'$
- p_s decides $\bar{\tau} \upharpoonright lh(s)$
- $(p_s, p_t) \in D_0 \cap D_1 \cap \ldots \cap D_i$ if $s, t \in 2^i$ and $s \neq t$

for $s, t \in 2^{<\omega}$.

Proof

- for $x \in 2^\omega$ let

$$g_x := \{q \in \bar{\mathbb{P}} : \exists n \in \omega \, p_{x|n} \leq q\}$$

- then g_x and g_y are mutually $\bar{\mathbb{P}}$-generic over $\bar{H}$ for $x \neq y$ so

$$\bar{H}[g_x, g_y] \models \neg \bar{\tau}^{g_x} E \bar{\tau}^{g_y}$$

- we have $\bar{H}[g_x, g_y] \prec_{\Sigma^1_{n+2}} V$ since $\bar{H}[g_x, g_y]$ computes $M_n^\#(z)$ correctly for each $z \in \mathbb{R} \cap \bar{H}[g_x, g_y]$.
- then $V \models \neg \bar{\tau}^{g_x} E \bar{\tau}^{g_y}$ since E is $\Pi^1_{n+3}(a)$
- now $\bar{\tau}^{g_x}$ depends continuously on x
- one would have a perfect set of pairwise inequivalent reals in V

Reasonable forcing and equivalence classes

Theorem

(S.) Suppose $M_n^\#(X)$ exists for every $X \in H_{\kappa^+}$ and is κ^+-iterable. Then reasonable forcing of size κ does not add equivalence classes to thin provably $\underset{\sim}{\Delta}^1_{n+3}$ equivalence relations.

Proof

- let E be a thin provably $\Delta^1_{n+3}(r)$ equivalence relation
- suppose a reasonable forcing $\mathbb{P}$ adds an equivalence class to E
- let τ be a $\mathbb{P}$-name for a real and $p \in \mathbb{P}$ a condition such that for every $x \in \mathbb{R}$ we have $p \Vdash_{\mathbb{P}} \neg \check{x} E \tau$
- Let $q \leq p$ be a condition with

$$(q, q) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau E \tau'$$

- since $\mathbb{P}$ is reasonable, there is a countable $H \prec V_\lambda$ with $r, \mathbb{P}, q, \tau \in H$ and λ sufficiently large, and a condition $r \leq q$, such that for every maximal antichain $A \subseteq \mathbb{P}$ with $A \in H$ the set $A \cap H$ is predense below q
- let $\pi : \bar{H} \to H$ be the uncollapsing map and $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{q}) = q$, $\pi(\bar{\tau}) = \tau$

Proof

- let g_0 be $\bar{\mathbb{P}}$-generic over $\bar{H}$ in V with $q \in g_0$
- let G be $\mathbb{P}$-generic over V with $r \in G$ and define $g_1 := \pi^{-1}{}''G$
- then $\bar{q} \in g_1$

Now g_1 is $\bar{\mathbb{P}}$-generic over $\bar{H}$:

- suppose $D \in \bar{H}$ is a dense subset of $\bar{\mathbb{P}}$
- then $H \models$ "$\pi(D)$ is dense in $\mathbb{P}$"
- $\pi(D) \cap H$ is predense below r since r is $(H, \mathbb{P})$-generic
- since $r \in G$ this implies $G \cap \pi(D) \cap H \neq \emptyset$
- so $g_1 \cap D \neq \emptyset$

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Proof

- now let h be $\bar{\mathbb{P}}$-generic over both $\bar{H}[g_0]$ and $\bar{H}[g_1]$ in V with $\bar{q} \in h$
- let $x_0 := \bar{\tau}^{g_0}$, $x_1 := \bar{\tau}^{g_1}$, and $y := \bar{\tau}^h$
- then $x_1 = \tau^G$
- we have $\bar{H}[g_0, h] \models x_0 E y$ and $\bar{H}[g_1, h] \models x_1 E y$ since $(\bar{q}, \bar{q}) \Vdash^{\bar{H}}_{\bar{\mathbb{P}} \times \bar{\mathbb{P}}} \bar{\tau} E \bar{\tau}'$
- $\bar{H}[g_i, h]$ calculates $M_n^\#(x)$ correctly by the previous lemma
- hence $\bar{H}[g_i, h] \prec_{\Sigma^1_{n+2}} V$
- then x_0, x_1, and y are E-equivalent since E is provably $\Delta^1_{n+3}(r)$
- but we assumed that $x_0 \in V$ and that x_1 is in a new equivalence class, a contradiction

Projective forcing

Definition

A forcing $(\mathbb{P}, \leq)$ is called a $\underset{\sim}{\Sigma}^1_n$ forcing if $\mathbb{P} \subseteq \mathbb{R}$ and both $\leq$ and $\perp$ are $\underset{\sim}{\Sigma}^1_n$ sets.

For example Cohen forcing, random forcing, and Amoeba forcing are $\underset{\sim}{\Sigma}^1_1$ (also called Suslin forcings) and also c.c.c.

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Results for Σ^1_2 forcing

We have analogues of the previous results for $\utilde{\Sigma}^1_2$ c.c.c. forcing.

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Lemma

(S.) Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Then Σ^1_{n+3}-absoluteness holds for every $\underset{\sim}{\Sigma}^1_2$ c.c.c. forcing $\mathbb{P}$.

Theorem

(S.) Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Then $\underset{\sim}{\Sigma}^1_2$ c.c.c. forcing does not add equivalence classes to thin provably $\underset{\sim}{\Delta}^1_{n+3}$ equivalence relations.

An open question

- can the results about thin equivalence relations and $\underset{\sim}{\Sigma}^1_2$ c.c.c. forcing be extended to projective c.c.c. forcing?
- thank you for listening!