MASS PROBLEMS

Stephen G. Simpson

Pennsylvania State University
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

Research supported by:
National Science Foundation of the USA,
grants DMS-0600823 and DMS-0652637,
Grove Endowment, Templeton Foundation

Logic Colloquium 2008

Bern, Switzerland

July 3–8, 2008

Introduction.

Algorithmically unsolvable problems occur frequently in mathematics and logic. Celebrated examples are:

- the halting problem for Turing machines,
- the word problem for groups,
- Hilbert's 10th problem for Diophantine equations,
- the Entscheidungsproblem for the predicate calculus,
- the problem of finding a completion of Peano arithmetic.

An important topic in mathematical logic dating back to the 1940s and 1950s is

degrees of unsolvability,

also known as *degree theory*. The purpose of degree theory is to classify algorithmically unsolvable problems according to the amount of unsolvability which is inherent in them.

Introduction (continued).

In the 1960s and 1970s, degree theory flourished. Remarkable contributions were made by Sacks, Lachlan, and many others.

In the 1980s and 1990s, degree theory fell into disrepute. In my opinion, this decline was due to an excessive concentration on methodological aspects, to the exclusion of foundationally significant aspects.

The purpose of this talk is to rehabilitate degree theory by means of a subtle but powerful shift in emphasis. The key concepts here are *mass problems* and *weak reducibility* a la Medvedev 1955 and Muchnik 1963.

Although known in Russia, mass problems were largely ignored in the West until recently. In this talk I hope to show that mass problems are an important link reconnecting degree theory to its roots in the foundations of mathematics.

Overview.

In this talk we survey some recent results on mass problems and their weak degrees.

Topics:

- mass problems
- weak reducibility
- weak degrees, a.k.a. Muchnik degrees
- connections with intuitionism
- $\mathcal{D}_w =$ the lattice of all weak degrees
- $\mathcal{P}_w =$ a countable sublattice of $\mathcal{D}_w$

Note: $\mathcal{P}_w$ consists of the weak degrees of nonempty, effectively closed sets of reals.

- some specific, natural degrees in $\mathcal{P}_w$
- randomness in $\mathcal{P}_w$
- hyperarithmeticity in $\mathcal{P}_w$
- Kolmogorov complexity in $\mathcal{P}_w$
- almost everywhere domination in $\mathcal{P}_w$
- an application of $\mathcal{P}_w$ to symbolic dynamics

Basic concepts.

By a *real* we mean a Turing oracle.

This could be either a real number, $x \in \mathbb{R}$,
or a point in the *Baire space*, $f \in \mathbb{N}^{\mathbb{N}}$,
or a point in the *Cantor space*, $X \in \{0,1\}^{\mathbb{N}}$.

Definitions (Medvedev 1955, Muchnik 1963)**.**

- A *mass problem* is a set of reals.
In other words, $P \subseteq \mathbb{R}$.

- The *solutions* of a mass problem P are the elements of P. In other words, $x \in P$.

- We say that *P is solvable* if there exists a solution $x \in P$ which is *Turing computable*.
Otherwise P is said to be *unsolvable*.

- Let P and Q be mass problems.
We say that *P is weakly reducible to Q*,
abbreviated $P \leq_w Q$, if
for all $y \in Q$ there exists $x \in P$ such that
x is computable using y as a *Turing oracle*.

Examples of unsolvable mass problems.

• Let CPA be the problem of finding a complete, consistent theory which extends Peano Arithmetic.

We identify CPA with a certain set of reals.

Namely, CPA =
$\{X \mid X$ is a completion of Peano Arithmetic$\}$
$= \{X \mid X$ is the characteristic function of the set of Gödel numbers of theorems of T, where T is a complete, consistent extension of Peano Arithmetic$\}$.

By Lindenbaum's Lemma, the set CPA is nonempty, i.e., there exist *solutions* of CPA.

On the other hand, by Gödel/Rosser/Tarski, CPA is an "unsolvable" problem, in the sense that there is no *computable* solution of CPA.

We may describe this situation by saying that the *degree of unsolvability* of CPA is greater than zero but less than infinity.

Examples (continued).

- Let R_1 be the problem of finding an infinite sequence of bits which is 1-*random* a la Martin-Löf 1966 and Kučera 1985. We identify R_1 with the set of 1-random points in the Cantor space.

It is known that R_1 is unsolvable, in the sense that no 1-random sequence of bits is Turing computable.

We can compare the degrees of unsolvability of these two unsolvable mass problems, CPA and R_1.

Namely, it can be shown that
R_1 is weakly reducible to CPA, but
CPA is not weakly reducible to R_1.

Thus, the degree of unsolvability of R_1 is less than that of CPA but greater than zero.

Intuitionistic motivation.

Kolmogorov 1932 proposed to view intuitionism as a "calculus of problems." This is the Brouwer/Heyting/Kolmogorov or BHK interpretation of intuitionism.

Using Turing's theory of computability, Medvedev 1955 and Muchnik 1963 gave rigorous elaborations of Kolmogorov's informal proposal.

A "problem" is defined instrumentally as the set of its possible solutions. The solutions are identified as reals. Thus, a mass problem is a set of reals.

A mass problem is defined to be "solvable" if at least one of its solutions is Turing computable.

According to Muchnik, a mass problem is "reducible" to another mass problem if, given any solution of the second problem, we can use it as a Turing oracle to compute a solution of the first problem.

Definitions (continued)**.**

- Two mass problems are said to be *weakly equivalent* if each is weakly reducible to the other. This is an equivalence relation.

- Let P be a mass problem.
The *weak degree of P* is the equivalence class of P under weak equivalence.
The weak degree of P is denoted $\deg_w(P)$.
Weak degrees are a.k.a. *Muchnik degrees*.

- $\mathcal{D}_w$ is the set of all Muchnik degrees, partially ordered by weak reducibility.
In other words, $\mathcal{D}_w = \{\deg_w(P) \mid P \subseteq \mathbb{R}\}$.
For $\mathbf{a} = \deg_w(P)$ and $\mathbf{b} = \deg_w(Q)$ we write $\mathbf{a} \leq \mathbf{b}$ if and only if $P \leq_w Q$.

Examples (continued)**.**

Note that $\mathbf{0} = \deg_w(\mathbb{R})$ is the Muchnik degree associated with *solvable* mass problems.
Letting $\mathbf{p} = \deg_w(\mathsf{CPA})$ and $\mathbf{r}_1 = \deg_w(\mathsf{R}_1)$, we have $\mathbf{0} < \mathbf{r}_1 < \mathbf{p}$ in $\mathcal{D}_w$.

Structural aspects of $\mathcal{D}_w$.

Theorem (Muchnik 1963)**.**
$\mathcal{D}_w$ is a complete distributive lattice.

Proof. For reals x and y say that
x is *Turing reducible to* y, abbreviated $x \leq_T y$,
if x is computable using y as a Turing oracle.

A set of reals U is said to be
Turing upward closed if
$x \in U$ and $x \leq_T y$ imply $y \in U$.

Note that $\mathcal{D}_w$ is dually isomorphic to the lattice of sets of reals which are Turing upward closed. The theorem follows.

Remark. The Turing upward closed sets are the open sets of *the Muchnik topology*. Muchnik explicitly noted that $\mathcal{D}_w$ is dually isomorphic to the lattice of open sets with respect to this topology.

Remark. Other structural aspects of $\mathcal{D}_w$ have been studied by Sorbi and Terwijn.

Applications to intuitionism.

As noted by Muchnik, the lattice $\mathcal{D}_w$ is *Brouwerian* in the sense of Birkhoff 1948. Thus $\mathcal{D}_w$ is a model of intuitionistic propositional calculus.

Recently we have been investigating *the Muchnik topos*, i.e., the category of sheaves of sets over the Muchnik space. It turns out that this topos contains some interesting models of intuitionistic analysis and set theory. Such models are similar to but simpler than the *relative realizability* models of Kleene/Vesley, J. Moschovakis, Troelstra, Birkedal, and van Oosten. For instance, we get a model of the scheme

$$(\forall f\, \exists g\, A(f,g)) \Rightarrow \exists h\, \forall f\, \exists g\, (g \leq_T f \oplus h \wedge A(f,g)).$$

This is work in progress.

These applications confirm the original insights of Kolmogorov 1932, Medvedev 1955, and Muchnik 1963.

The sublattice $\mathcal{P}_w$.

Remark. $\mathcal{D}_w$ is of cardinality $2^{2^{\aleph_0}}$. We now focus on a countable sublattice of $\mathcal{D}_w$ which was introduced by me in 1999.

Definitions.

- A set of reals is *effectively closed* if it is the complement of the union of a computable sequence of basic open sets. Effectively closed sets are also known as Π^0_1 sets.

- $\mathcal{P}_w$ is the set of weak degrees of mass problems associated with nonempty, effectively closed sets in

  the Cantor space, $\{0,1\}^{\mathbb{N}}$,

  partially ordered by weak reducibility.

Remark. $\mathcal{P}_w$ is unaffected if we replace the Cantor space $\{0,1\}^{\mathbb{N}}$ by the real line, $\mathbb{R}$, or n-dimensional Euclidean space, $\mathbb{R}^n$. However, Π^0_1 sets in the *Baire space* $\mathbb{N}^{\mathbb{N}}$ behave differently, because of the lack of local compactness.

First results on $\mathcal{P}_w$.

- $\mathcal{P}_w$ is a countable distributive lattice. It has a bottom element, $\mathbf{0} = \deg_w(\mathbb{R})$. Moreover $\mathbf{0}$ is meet-irreducible. (These results are easy.)

- $\mathcal{P}_w$ has a top element, $\mathbf{1} = \deg_w(\mathsf{CPA})$. (Scott/Tennenbaum 1960.)

- The degree $\mathbf{r}_1 = \deg_w(\mathsf{R}_1)$ belongs to $\mathcal{P}_w$. Within $\mathcal{P}_w$ we have $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$ and $\mathbf{r}_1$ is meet-irreducible and does not join to $\mathbf{1}$. Moreover $\mathbf{r}_1$ can be characterized as the maximum weak degree of an effectively closed set of positive measure. (Kučera 1985, Simpson 1999.)

- Every countable distributive lattice is lattice-embeddable in every nontrivial initial segment of $\mathcal{P}_w$. (Binns/Simpson 2004.)

- Every element of $\mathcal{P}_w$ is join-reducible. (Binns 2003.)

- We conjecture that for all $\mathbf{a} < \mathbf{b}$ in $\mathcal{P}_w$ there exists $\mathbf{c}$ in $\mathcal{P}_w$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. This would be analogous to the Sacks Density Theorem for r.e. Turing degrees.

Specific degrees in $\mathcal{P}_w$.

In addition to being structurally rich, $\mathcal{P}_w$ contains many specific, natural degrees which are of interest from a computational viewpoint. These degrees tend to be linked to foundationally significant topics such as:

- algorithmic randomness
- effective Hausdorff dimension
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- the hyperarithmetical hierarchy
- resource-bounded computational complexity
- Kolmogorov complexity
- subrecursive hierarchies

We shall define and explain some of these specific, natural degrees in $\mathcal{P}_w$.

First we need a technical lemma.

The Embedding Lemma.

Many advanced results concerning $\mathcal{P}_w$ are based on the following lemma.

Notation. We use sup and inf to denote the supremum and infimum operations in $\mathcal{D}_w$. Thus sup = join = least upper bound, and inf = meet = greatest lower bound.

Remark. For $\mathbf{a} = \text{deg}_w(P)$ and $\mathbf{b} = \text{deg}_w(Q)$ we have $\sup(\mathbf{a}, \mathbf{b}) = \text{deg}_w(P \times Q)$ and $\inf(\mathbf{a}, \mathbf{b}) = \text{deg}_w(P \cup Q)$.

Definition. Let S be a set of reals. S is said to be Σ^0_3 if

$$S = \{X \mid \exists i \, \forall m \, \exists n \, R(X, i, m, n)\}$$

where $R \subseteq \{\text{reals}\} \times \mathbb{N}^3$ is recursive.

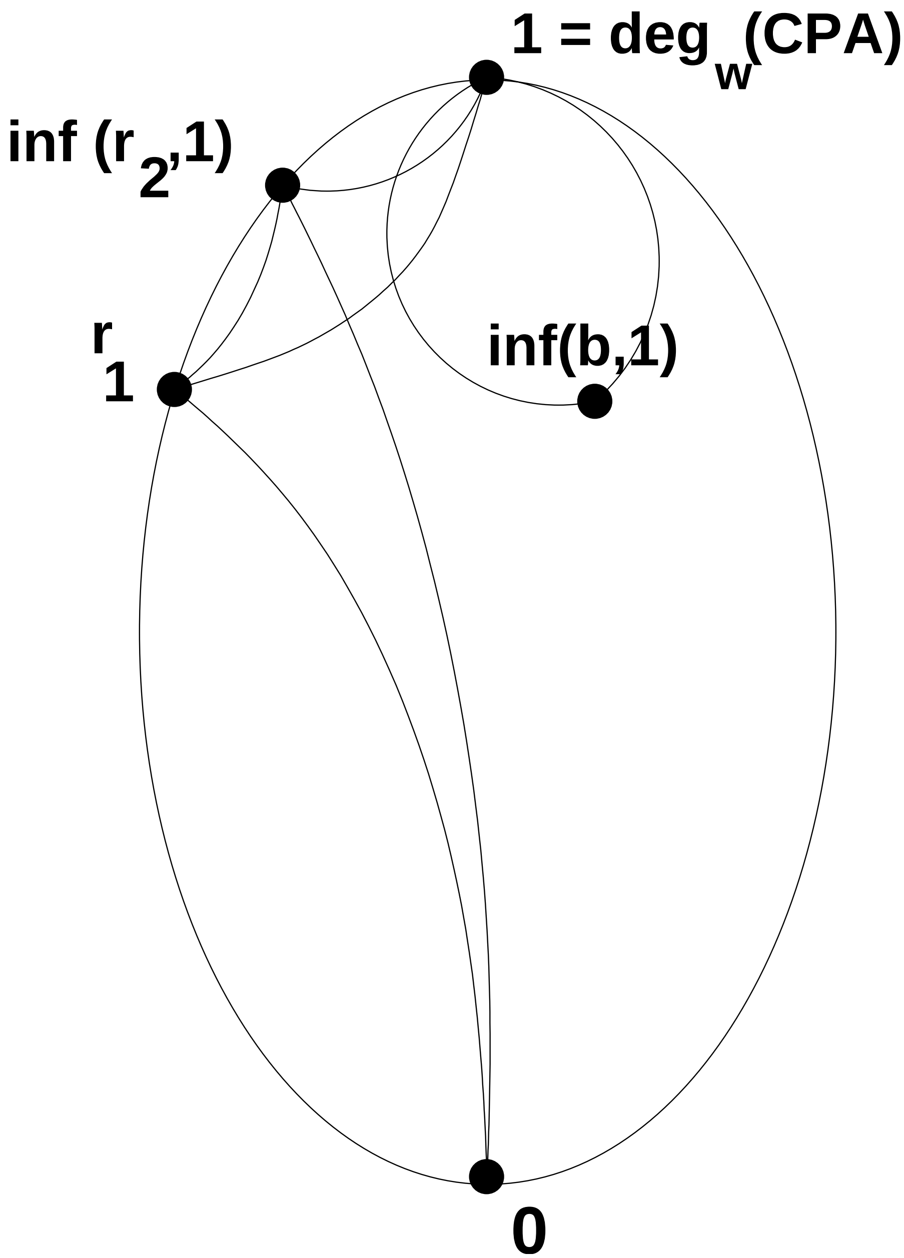
Embedding Lemma (Simpson 2003)**.** Let $\mathbf{s} = \text{deg}_w(S)$ where S is Σ^0_3. Then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{P}_w$.

Applications of the Embedding Lemma.

- Let R_2 be the set of reals which are <u>*2-random*</u>, i.e., 1-random relative to the Halting Problem. It can be shown that R_2 is Σ^0_3. Therefore, letting $\mathbf{r}_2 = \deg_w(\mathsf{R}_2)$ we have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{P}_w$. Moreover $\inf(\mathbf{r}_2, \mathbf{1})$ is meet-irreducible and does not join to $\mathbf{1}$ and is the maximum weak degree of a Π^0_1 set whose Turing upward closure is of positive measure. (Simpson 2003.)

- Let X and Y be reals. We say that <u>*X is dominated by Y*</u> if for all $f \leq_T X$ there exists $g \leq_T Y$ such that $f(n) < g(n)$ for all n. A real is <u>*almost everywhere dominating*</u> if it dominates all reals except a set of measure 0. Let $\mathsf{AED} = \{Y \mid Y$ is almost everywhere dominating$\}$. It can be shown that AED is Σ^0_3. Therefore, letting $\mathbf{b} = \deg_w(\mathsf{AED})$ we have $\inf(\mathbf{b}, \mathbf{1}) \in \mathcal{P}_w$.

(Binns, Cholak, Dobrinen, Greenberg, Kjos-Hanssen, Lerman, Miller, Simpson, Solomon, 2004–2006.)



A picture of $\mathcal{P}_w$. Here $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{b}$ are the weak degrees associated with 1-randomness, 2-randomness, and almost everywhere domination.

Applications (continued).

- The lattice $\mathcal{P}_w$ is not Brouwerian. This is proved using the Embedding Lemma plus my generalization of the Posner/Robinson Theorem. (Simpson 2007.)

- Recall that $\varphi_n^{(1)}(n)$ is a universal partial recursive function. We say that $f \in \mathbb{N}^{\mathbb{N}}$ is <u>*diagonally nonrecursive*</u> if $\forall n\,(f(n) \neq \varphi_n^{(1)}(n))$. Let $\mathbf{d} = \deg_w(\text{DNR})$ where DNR is the set of functions which are diagonally nonrecursive. We can use the Embedding Lemma to show that $\mathbf{d}$ belongs to $\mathcal{P}_w$. (Simpson 2003.)

- A function $f \in \mathbb{N}^{\mathbb{N}}$ is <u>*recursively bounded*</u> if there exists a recursive function $g \in \mathbb{N}^{\mathbb{N}}$ such that $f(n) < g(n)$ for all n. Let DNR_{REC} be the set of functions which are diagonally nonrecursive and recursively bounded.
Let $\mathbf{d}_{\text{REC}} = \deg_w(\text{DNR}_{\text{REC}})$.
We can use the Embedding Lemma to show that $\mathbf{d}_{\text{REC}}$ belongs to $\mathcal{P}_w$. (Simpson 2003.)

Applications (continued).

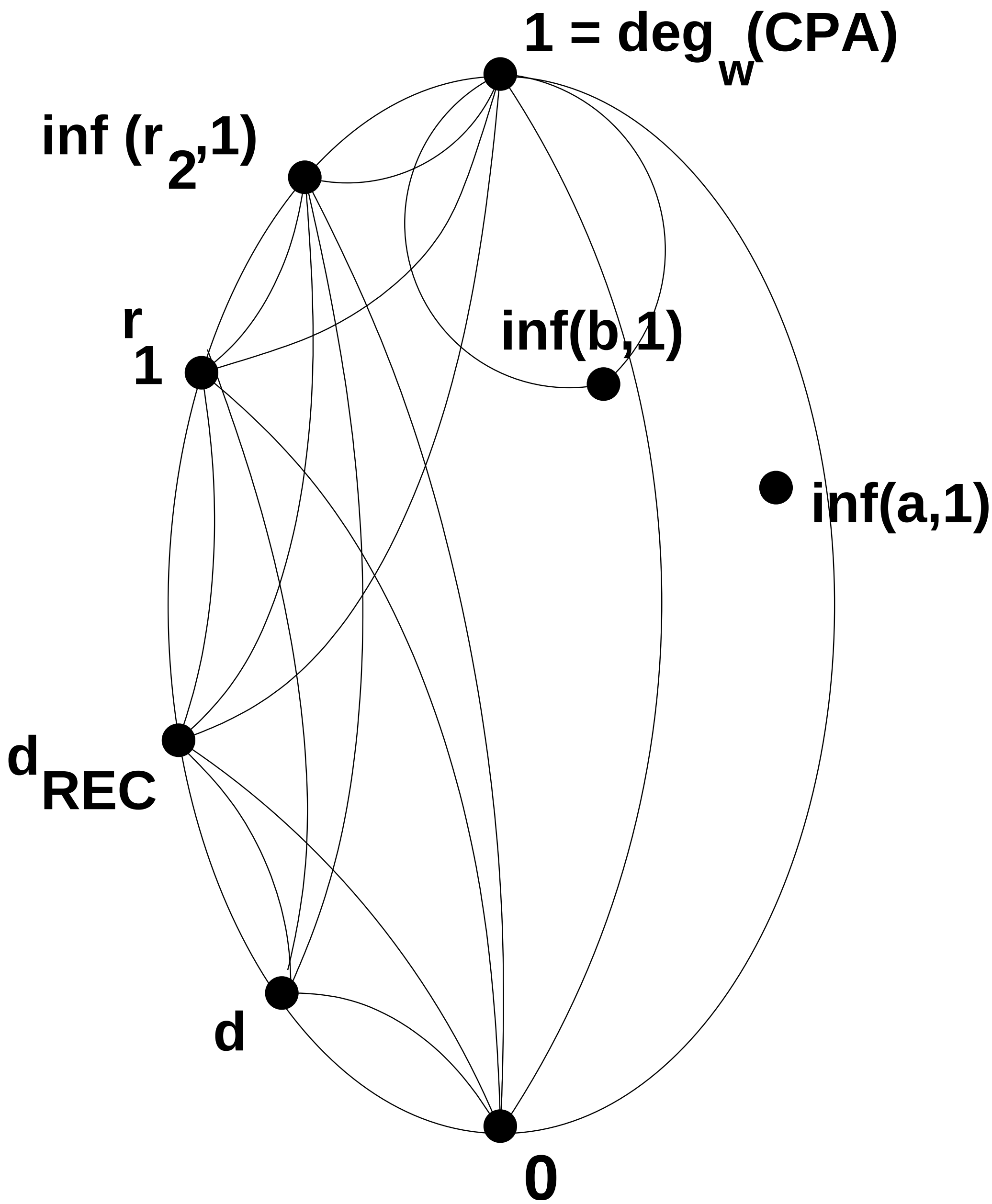
• Given a recursively enumerable set $A \subseteq \mathbb{N}$, map the Turing degree of A to the Muchnik degree of $\mathsf{CPA} \cup \{A\}$. By the Embedding Lemma, this Muchnik degree belongs to $\mathcal{P}_w$. Thus we have a natural embedding of all such Turing degrees into $\mathcal{P}_w$. A result of Arslanov implies that this embedding is one-to-one and preserves the top and bottom and upper semilattice structure of the recursively enumerable Turing degrees. (Simpson 2003.)

Remark. There is an obvious analogy between $\mathcal{P}_w$ and the upper semilattice of recursively enumerable Turing degrees.

However, $\mathcal{P}_w$ is much better, because we know many specific examples of natural, intermediate degrees in $\mathcal{P}_w$. For instance

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1}\,.$$

No such examples are known in the case of the recursively enumerable Turing degrees.



A picture of $\mathcal{P}_w$. Here $\mathbf{r} =$ randomness,
$\mathbf{b} =$ almost everywhere domination,
$\mathbf{d} =$ diagonal nonrecursiveness,
$\mathbf{a} =$ any recursively enumerable degree.

Embedding hyperarithmeticity into $\mathcal{P}_w$.

Recall that a *recursive ordinal* is the order type of a recursive well ordering of a set of integers. For each such ordinal α, let $0^{(\alpha)}$ be the αth Turing jump of 0. This is well defined up to Turing degree. These Turing degrees are known as *the hyperarithmetical hierarchy*.

Recently Cole/Simpson 2006 exhibited a natural embedding of the hyperarithmetical hierarchy into $\mathcal{P}_w$. We now outline this result.

Let $\mathbf{h}_\alpha$ be the Muchnik degree of $0^{(\alpha)}$. The Embedding Lemma implies that $\inf(\mathbf{h}_\alpha, \mathbf{1})$ belongs to $\mathcal{P}_w$, but this is worthless because $\inf(\mathbf{h}_\alpha, \mathbf{1}) = \mathbf{1}$. The Cole/Simpson embedding is slightly more complicated.

Embedding hyperarithmeticity (continued).

Definitions (Cole/Simpson 2006).

Let X be a real. A function $f(n)$ is called <u>*boundedly limit recursive in* X</u> if there exist an X-recursive approximating function $\widetilde{f}(n,s)$ and a recursive bounding function $\widehat{f}(n)$ such that for all n, $f(n) = \lim_s \widetilde{f}(n,s)$ and $|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| < \widehat{f}(n)$.

Let $\mathsf{BLR}(X) = \{f \mid f$ is boundedly limit recursive in $X\}$. If S is a set of reals, let

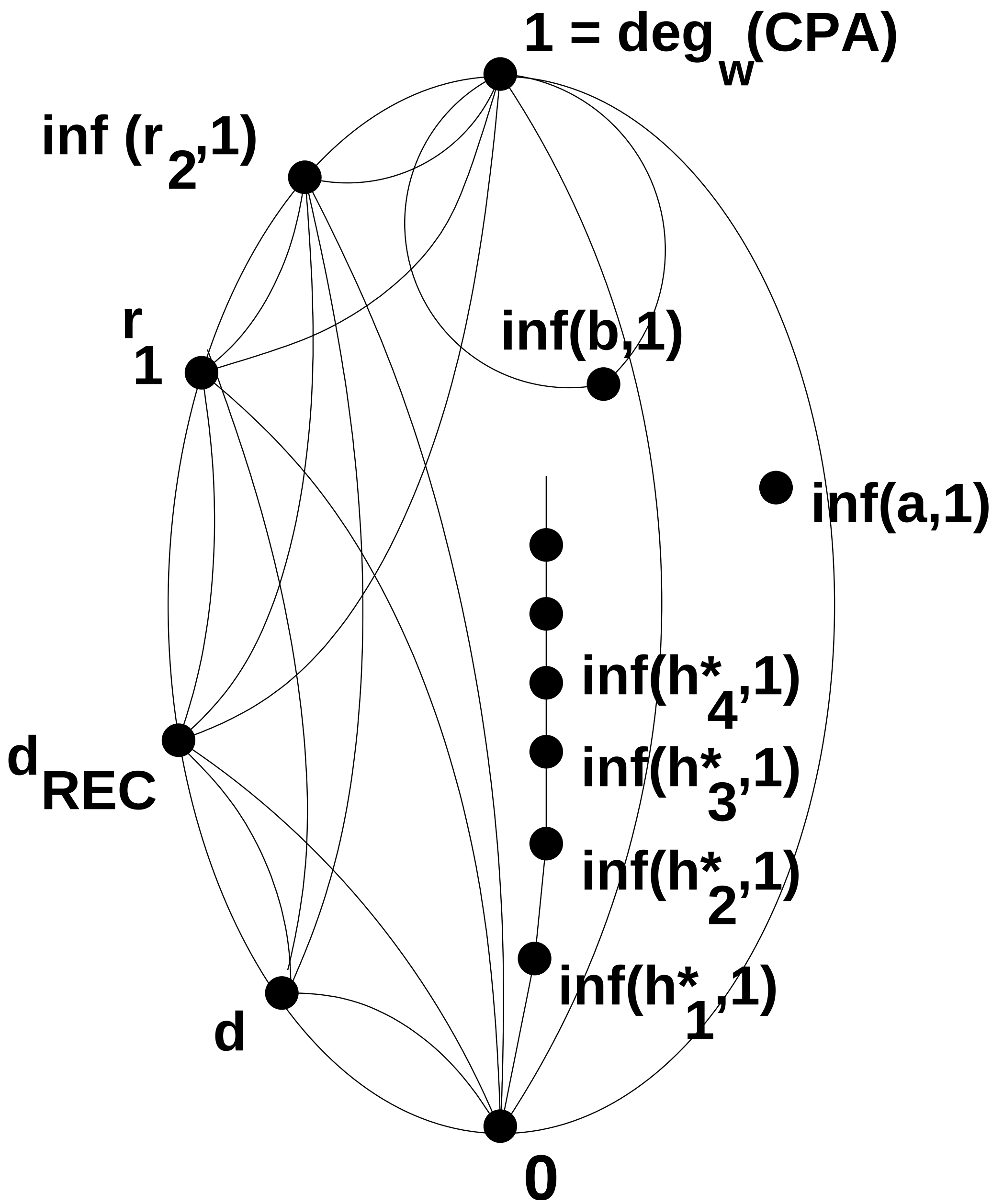
$S^* = \{Y \mid \exists X\,(X \in S \wedge \mathsf{BLR}(X) \subseteq \mathsf{BLR}(Y))\}$.

If $\mathbf{s} = \deg_w(S)$ let $\mathbf{s}^* = \deg_w(S^*)$.

This is well defined up to Muchnik degree.

Theorems (Cole/Simpson 2006).

If S is Σ^0_3 then S^* is Σ^0_3. For all recursive ordinals $0 < \alpha < \beta$ we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{P}_w$ and $\mathbf{0} < \inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_\beta, \mathbf{1}) < \mathbf{1}$ and they are incomparable with $\mathbf{d}$ and $\inf(\mathbf{r}_2, \mathbf{1})$.



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ almost everywhere domination, $\mathbf{d} =$ diagonal nonrecursiveness.

In the above picture, each of the black dots except the one labeled $\inf(\mathbf{a}, \mathbf{1})$ represents a specific, natural Muchnik degree in $\mathcal{P}_w$.

We shall now exhibit some more black dots.

Kolmogorov complexity.

Definition. Given $X \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

let $X \upharpoonright n = \langle X(0), X(1), \ldots, X(n-1) \rangle$.

Let $K(X \upharpoonright n) =$ the *Kolmogorov/Chaitin complexity of* $X \upharpoonright n$, i.e., the minimum size (measured in bits) of a program (prefix-free) which describes $X \upharpoonright n$.

Remark. It is known that X is 1-random if and only if $\exists c \forall n \, (K(X \upharpoonright n) \geq n - c)$.

We consider two refinements.

- The *effective Hausdorff dimension of X* is defined as $\dim(X) = \liminf_n K(X \upharpoonright n)/n$. For right r.e. real numbers $0 \leq s < 1$ let $\mathbf{q}_s$ be the Muchnik degree of $\{X \mid \dim(X) > s\}$. It can be shown that $\mathbf{q}_s \in \mathcal{P}_w$. Moreover $s < t$ implies $\mathbf{q}_s < \mathbf{q}_t$. (J. Miller 2008.)

Kolmogorov complexity (continued).

• Let C be a "nice" class of recursive functions. For example, C could be the polynomial time computable functions, or the primitive recursive functions, or all recursive functions.

Define $X \in \{0,1\}^{\mathbb{N}}$ to be <u>*C-complex*</u>

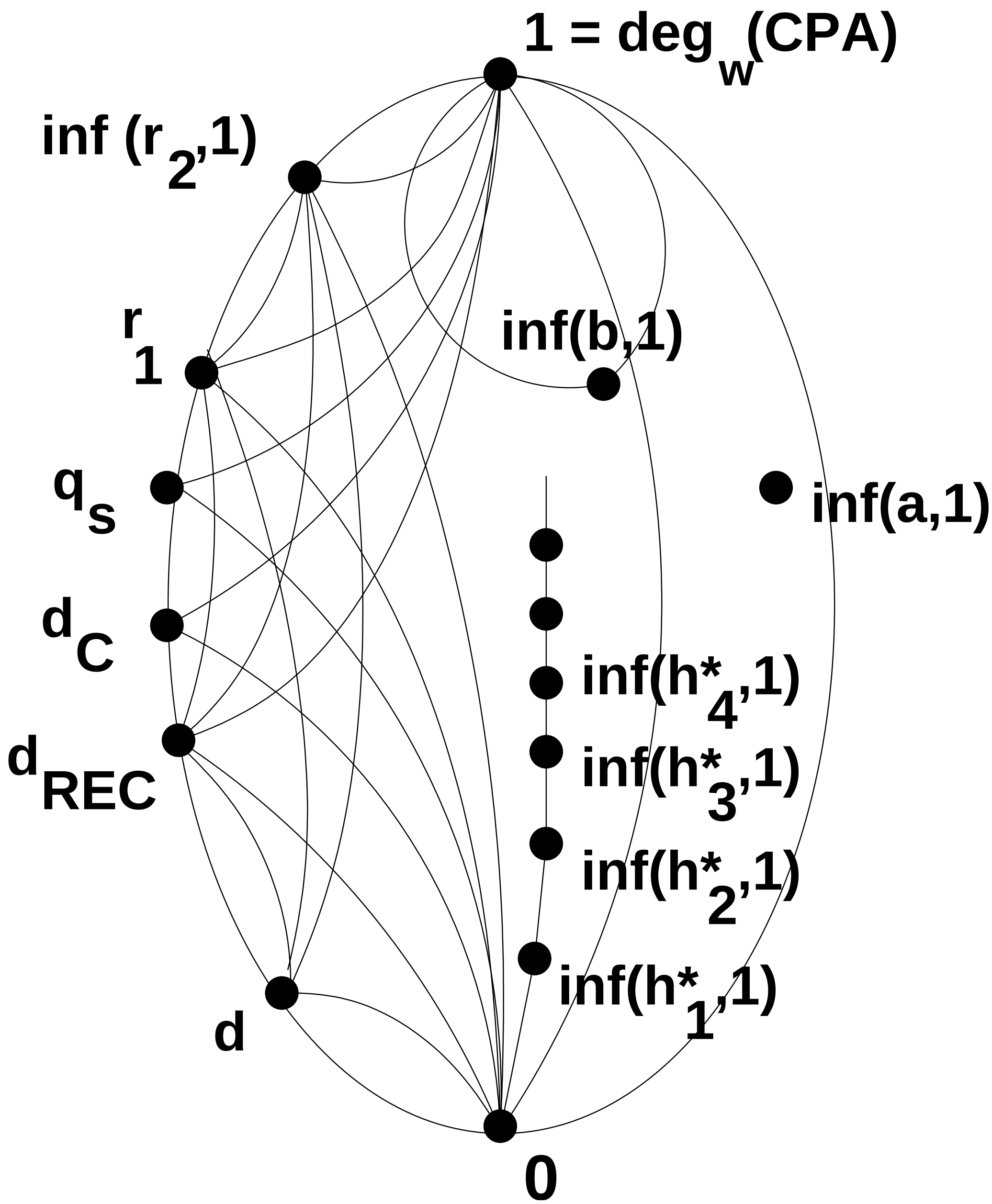
if $(\exists f \in C)\,(\forall n)\,(K(X \upharpoonright n) \geq f^{-1}(n))$.

Here $f^{-1}(n) =$ least m such that $f(m) \geq n$.

It turns out that the problem of finding a C-complex real is weakly equivalent to the problem of finding a function which is diagonally nonrecursive and C-bounded.

Moreover, letting $\mathbf{d}_C$ be the Muchnik degree of this problem, we have $\mathbf{d}_C \in \mathcal{P}_w$. Moreover, if C' is another such class and contains a function which grows "much faster than" all functions in C, then $\mathbf{d}_C < \mathbf{d}_{C'}$.

(Ambos-Spies, Kjos-Hanssen, Lempp, Merkle, Simpson, Slaman, Stephan, 2004–2006.)



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ a. e. domination, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

Application to symbolic dynamics.

A *tiling problem* is a finite set of unit squares with colored edges. The elements of the set are called *tiles*. A *solution* of the problem is an assignment of tiles to integer points in the plane so that adjacent edges have matching colors. Tilings of the plane were studied in the 1960s and 1970s by logicians including Wang, R. Berger, R. Robinson, and Myers. For instance, it is undecidable whether the solution set of a given tiling problem is nonempty.

Recently a link with symbolic dynamics has emerged. Namely, nonempty solution sets of tiling problems are essentially the same thing as 2-dimensional subshifts of finite type. See for instance the recent theorem of Hochman/Meyerovitch saying that a positive real number is the entropy of such a subshift if and only if it is right recursively enumerable.

Symbolic dynamics (continued)**.**

Using the methods of Robinson and Myers, we have proved:

Theorem (Simpson 2007)**.**
$\mathcal{P}_w$ consists precisely of the Muchnik degrees of 2-dimensional subshifts of finite type.

This theorem has consequences in symbolic dynamics. For instance, using this theorem together with structural properties of $\mathcal{P}_w$, we obtain an infinite sequence of 2-dimensional subshifts of finite type which are strongly independent of each other.

Surely there are correlations between dynamical properties of 2-dimensional subshifts of finite type and computational properties of their Muchnik degrees. This aspect remains largely unexplored.

Some references.

Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman. Comparing DNR and WWKL. *Journal of Symbolic Logic*, 69, 2004, 1089–1104.

Stephen Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49, 2003, 327–335.

Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43, 2004, 399–414.

Joshua A. Cole and Stephen G. Simpson. Mass problems and hyperarithmeticity. *Journal of Mathematical Logic*, 7, 2008, 125–143.

Natasha L. Dobrinen and Stephen G. Simpson. Almost everywhere domination. *Journal of Symbolic Logic*, 69, 2004, 914–922.

Bjørn Kjos-Hanssen. Low for random reals and positive-measure domination. *Proceedings of the American Mathematical Society*, 135, 2007, 3703–3709.

Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the recursion theorem. *STACS 2006 Proceedings*, Springer Lecture Notes in Computer Science, 3884, 2006, 149–161.

Some references (continued).

Bjørn Kjos-Hanssen and Stephen G. Simpson. Mass problems and Kolmogorov complexity. Preprint, October 2006, in preparation.

Bjørn Kjos-Hanssen, Joseph S. Miller, and David Reed Solomon. Lowness notions, measure and domination. Preprint, 20 pages, May 2008, in preparation.

Andrei N. Kolmogorov. Zur Deutung der intuitionistischen Logik. *Mathematische Zeitschrift*, 35, 1932, 58–65.

Yuri T. Medvedev. Degrees of difficulty of mass problems. *Doklady Akademii Nauk SSSR*, 104, 1955, 501–504, in Russian.

Joseph S. Miller. Extracting information is hard. Preprint, 11 pages, May 2008, to appear.

Albert A. Muchnik. On strong and weak reducibilities of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4, 1963, 1328–1341, in Russian.

Stephen G. Simpson. FOM list, August 13, 1999.

Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11, 2005, 1–27.

Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*, 75, 2007, 287–297.

Some references (continued)**.**

Stephen G. Simpson. Almost everywhere domination and superhighness. *Mathematical Logic Quarterly*, 53, 2007, 462–482.

Stephen G. Simpson. Mass problems and almost everywhere domination. *Mathematical Logic Quarterly*, 53, 2007, 483–492.

Stephen G. Simpson. Medvedev degrees of 2-dimensional subshifts of finite type. Preprint, 8 pages, May 2007, *Ergodic Theory and Dynamical Systems*, to appear.

Stephen G. Simpson. Mass problems and intuitionism. *Notre Dame Journal of Formal Logic*, 49, 2008, 127–136.

Stephen G. Simpson. Some fundamental issues concerning degrees of unsolvability. *Computational Prospects of Infinity, Part II: Presented Talks*, National University of Singapore, Lecture Notes Series, Number 15, World Scientific, 2008, 313–332.

Stephen G. Simpson. The Muchnik topos. 2008, in preparation.

THE END