Positive definite functions on equivalence relations

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joint work with A. Ioana and A. S. Kechris

Definable equivalence relations

Definable equivalence relations on standard Borel spaces have been studied extensively by descriptive set theorists during the last few decades (Jackson, Hjorth, Kechris, Louveau, ...).

An interesting feature of the subject is its close interaction with other areas of mathematics. For example, the study of the orbit equivalence relations of Polish group actions uses tools from topological dynamics, while countable equivalence relations are intimately connected with ergodic theory.

Measured equivalence relations

- (X, μ) is a standard probability space (X is a standard Borel space and μ a non-atomic, Borel, probability measure ($[0,1]$, Lebesgue measure)).
- An equivalence relation E on X is **Borel** if $E \subseteq X^2$ is Borel and **countable** if all equivalence classes are countable.
- The orbit equivalence relation of a group action $\Gamma \curvearrowright X$ is defined by
$$x_1 \, E_\Gamma^X \, x_2 \iff \exists \gamma \in \Gamma \quad \gamma \cdot x_1 = x_2.$$
- Every countable Borel equivalence relation is the orbit equivalence relation of a Borel action of a countable group (Feldman and Moore).
- E is called **measure-preserving** if it is generated by a measure-preserving group action. It is **ergodic** if every E-invariant set is either null or conull.

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- Null sets will be routinely ignored and all statements hold only almost everywhere.

Orbit equivalence

Definition

Two measure-preserving equivalence relation E and F on spaces (X, μ) and (Y, ν), respectively are **isomorphic** if there are invariant Borel sets $A \subseteq X$ and $B \subseteq Y$ of full measure and a measure-preserving map $f \colon A \to B$ such that

$$x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2).$$

Two measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (Y, \nu)$ are called **orbit equivalent** if their orbit equivalence relations E_Γ^X and E_Δ^Y are isomorphic.

Orbit equivalence has become an important meeting point of ergodic theory, Borel equivalence relations, and von Neumann algebras.

Amenable groups I

Theorem (Dye, 1963)

All ergodic actions of $\mathbf{Z}$ are orbit equivalent.

An equivalence relation generated by a $\mathbf{Z}$ action is called **hyperfinite**.

Theorem (Ornstein–Weiss, 1980)

If $\Gamma \curvearrowright X$ is measure-preserving and Γ is amenable, then E_Γ^X is hyperfinite a.e.

Corollary

All ergodic actions of amenable groups are orbit equivalent.

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Examples: $\mathbf{Z}$, solvable groups, groups with polynomial growth, etc. The class of amenable groups is closed under taking subgroups and factors.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbf{N}$; define a finitely additive, invariant probability measure m on $\mathbf{Z}$ by:

$$m(A) = \lim_{n\to\mathcal{U}} \frac{|A \cap [-n,n]|}{2n+1}.$$

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Non-examples: non-abelian free groups.

Non-orbit equivalent actions of non-amenable groups

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Continuum many non-orbit equivalent, ergodic actions for:

- property (T) groups (Hjorth);
- non-abelian free groups (using relative property (T) of the pair $(\mathrm{SL}(2,\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$) (Gaboriau–Popa);
- groups containing $\mathbf{F}_2$ (Ioana);
- all non-amenable groups (Epstein).

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Orbit equivalence of ergodic actions of $\mathbf{F}_2$ is:

- not smooth (one cannot use reals as invariants of OE) (Törnquist);
- not classifiable by countable structures (Kechris, Törnquist).

Non-classifiability by countable structures

Let $\mathcal{L} = \langle R_1, R_2, \dots, f_1, f_2, \dots \rangle$ be a countable language. Then one can consider the Polish space of all countable (infinite) $\mathcal{L}$-structures $X_{\mathcal{L}}$. For example, if $\mathcal{L} = \langle R, f \rangle$, where R has arity l and f has arity k, every countable $\mathcal{L}$-structure with universe $\mathbf{N}$ can be viewed as an element of $2^{\mathbf{N}^l} \times \mathbf{N}^{\mathbf{N}^k}$.

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Countable structures can be used for classification as follows:

Definition

We say that an equivalence relation E on a standard Borel space X **can be classified by countable structures** if there exists a language $\mathcal{L}$ and a Borel map $f \colon X \to X_{\mathcal{L}}$ such that

$$x \, E \, y \iff f(x) \cong f(y).$$

The main tool for proving non-classifiability by countable structures is Hjorth's theory of turbulence.

Epstein's co-inducing construction

- Given $\Delta \leq \Gamma$ and an action $\Delta \curvearrowright (Y, \nu)$, one can define an action

$$\Gamma \curvearrowright \{f\colon \Gamma \to Y : f \text{ is } \Delta\text{-equivariant}\} \cong Y^{\Gamma/\Delta} \quad \text{by}$$
$$(\gamma \cdot f)(\gamma') = f(\gamma^{-1}\gamma').$$

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- If instead one is given two actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (X, \mu)$ such that $E_\Delta^X \subseteq E_\Gamma^X$, one can define from an action $\Delta \curvearrowright (Y, \nu)$ another action

$$\Gamma \curvearrowright \{(x, f) : f\colon [x]_\Gamma \to Y, f \text{ is } \Delta\text{-equivariant}\} \cong X \times Y^{[E:F]}.$$

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- If instead one is given two actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (X, \mu)$ such that $E^X_\Delta \subseteq E^X_\Gamma$, one can define from an action $\Delta \curvearrowright (Y, \nu)$ another action

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- Gaboriau and Lyons proved (using percolation) that for any non-amenable group Γ, the Bernoulli shift $\Gamma \curvearrowright [0,1]^\Gamma$ admits a subequivalence relation generated by a free, ergodic action of $\mathbf{F}_2$.

Positive definite functions

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Definition

Let Γ be a countable group. A function $\phi\colon \Gamma \to \mathbf{C}$ is called **positive definite** if for any $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbf{C}$,

$$\sum_{i,j} \phi(\gamma_i^{-1}\gamma_j) c_i \overline{c_j} \geq 0.$$

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Let $F \subseteq E$ be equivalence relations on (X, μ) and E be generated by an action of Γ. Then the function

$$\tau_F(\gamma) = \mu(\{x : \gamma \cdot x \mathrel{F} x\})$$

is positive definite.

The function τ_F can be used to measure how big F is inside E.

Generating mixing actions

Definition

An action $\Gamma \curvearrowright (X, \mu)$ is called **mixing** if for any two measurable sets $A, B \subseteq X$,

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B).$$

Easily from the definition, all mixing actions are ergodic.

$\Gamma, \Delta \curvearrowright (X, \mu)$ are fixed; $E^X_\Delta \subseteq E^X_\Gamma$. Write $F = E^X_\Delta$.

Proposition

If $\Gamma \curvearrowright (X, \mu)$ is mixing and $\tau_F(\gamma) \to 0$ as $\gamma \to \infty$, then for any action $\Delta \curvearrowright (Y, \nu)$, the co-induced action $\Gamma \curvearrowright X \times Y^{[E:F]}$ is mixing.

Existence of small subequivalence relations

Theorem

Let Γ be a countable group and $\Gamma \curvearrowright (X, \mu)$ a mixing action. Then there exists an ergodic, hyperfinite subequivalence relation $F \subseteq E_\Gamma^X$ such that $\tau_F(\gamma) \to 0$ as $\gamma \to \infty$.

This theorem, together with Epstein's construction and the previous proposition, provides many new mixing actions of arbitrary groups.

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Theorem (with I. Epstein)

Let Γ be non-amenable. Then there exists a free, measure-preserving, mixing action $\Gamma \curvearrowright (X, \mu)$ and a measure-preserving, ergodic action $\mathbf{F}_2 \curvearrowright (X, \mu)$ such that $E_{\mathbf{F}_2}^X \subseteq E_\Gamma^X$ and $\tau_{E_{\mathbf{F}_2}^X}(\gamma) \to 0$ as $\gamma \to \infty$.

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Theorem (with I. Epstein)

Let Γ be a countable, non-amenable group. Then orbit equivalence of free, mixing actions of Γ is not classifiable by countable structures.

Property (T) groups

Definition

A group Γ has **property (T)** if there exist a finite set $Q \subseteq \Gamma$ and $\epsilon > 0$ such that every unitary representation of Γ that has a (Q, ϵ)-almost invariant unit vector actually has a fixed point. More precisely, for every unitary representation π on a Hilbert space $\mathcal{H}$, if there exists a unit vector $\xi \in \mathcal{H}$ for which $\|\pi(\gamma)(\xi) - \xi\| < \epsilon$ for all $\gamma \in Q$, then there is $0 \neq \xi_0 \in \mathcal{H}$ such that $\pi(\gamma)(\xi_0) = \xi_0$ for all $\gamma \in \Gamma$. (Q, ϵ) is called a **Kazhdan pair** for Γ.

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- Examples: $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$;
- Property (T) is incompatible with amenability: every amenable group with property (T) is finite.

Closeness of subequivalence relations

Theorem

Let $\Gamma \curvearrowright (X, \mu)$ be measure-preserving and let $F \subseteq E_\Gamma^X$ be a subequivalence relation. If

$$\inf_{\gamma \in \Gamma} \tau_F(\gamma) = \tau_0 > 0,$$

then there exists an F-invariant set A of positive measure such that

$$[E_\Gamma^X | A : F | A] \leq \frac{1}{\tau_0}.$$

In particular, if F is ergodic, it has finite index in E_Γ^X.

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Example (Property (T) groups)

Let Γ have property (T) and (Q, ϵ) be a Kazhdan pair for Γ. If for all $\gamma \in Q$, $\tau_F(\gamma) > 1 - \epsilon^2/2$, then the conclusion of the theorem holds.

Let Γ be a finitely generated group with a finite, symmetric generating set S. Let $\mathcal{G}$ be the (right) Cayley graph of Γ (the vertices of the graph are the elements of Γ and we connect γ and γ' with an edge if $\gamma s = \gamma'$ for some $s \in S$). Let E be the set of edges of $\mathcal{G}$.

Percolation

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Definition

A **percolation** on the graph $\mathcal{G}$ is a Γ-invariant measure $\mathbf{P}$ on 2^{E}.
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Percolation theory, especially on $\mathbf{Z}^d$ but, more recently, also on arbitrary Cayley graphs, is a well developed branch of probability theory.

Basic facts about Bernoulli percolation

The main question in percolation theory is about the existence of infinite clusters (connected components) of the (random) percolation graph. It is easy to see that the number of infinite clusters in Bernoulli percolation is deterministic (constant a.s.).

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Basic facts:

- The number of infinite clusters is always 0, 1, or ∞.
- There exist constants p_c and p_u such that $0 < p_c \le p_u \le 1$ and the following hold:
  - if $p \in [0, p_c)$, there are no infinite clusters;
  - if $p \in (p_c, p_u)$, there are infinitely many infinite clusters;
  - if $p \in (p_u, 1]$, there is a unique infinite cluster.
- If Γ is amenable, then $p_c = p_u$ (Burton and Keane).
- **Conjecture** (Benjamini–Schramm): If Γ is non-amenable, then $p_c < p_u$.

Percolation as a source of subequivalence relations

Consider the natural action $\Gamma \curvearrowright 2^{\mathsf{E}}$ by shift. Let E be the orbit equivalence relation. One can define the cluster subequivalence relation $E_c \subseteq E$ as follows:

$$\omega_1 \, E_c \, \omega_2 \iff \exists \gamma \in \Gamma \quad \gamma \cdot \omega_1 = \omega_2 \text{ and } 1 \text{ and } \gamma \text{ are in the same } \omega_1\text{-cluster.}$$

In this way, every E-equivalence class gets the structure of a subgraph of the Cayley graph $\mathcal{G}$ and the equivalence classes of E_c are just the connected components of this subgraph.

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Notice that in this case, the function τ_{E_c} has a very simple interpretation, namely:

$$\tau_{E_c}(\gamma) = \mathbf{P}(1 \text{ and } \gamma \text{ are in the same cluster}).$$

In particular, if $\gamma \in S$, then

$$\tau_{E_c}(\gamma) \geq p.$$

An application to percolation on property (T) groups

Conjecture (Benjamini–Schramm)

Let $\mathcal{G}$ be a Cayley graph of Γ. Then $p_u < 1$ iff Γ has one end ($\mathcal{G} \setminus A$ has only one infinite connected component for all finite A).

Theorem (Lyons–Schramm)

If Γ has property (T), then $p_u < 1$.

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Theorem (Lyons–Schramm)

If Γ has property (T), then $p_u < 1$.

Theorem

Let Γ have property (T) with Kazhdan pair (Q, ϵ). Then $p_u(\mathcal{G}_Q) \leq 1 - \epsilon^2/2$.