A Cut Free Axiomatization for Relativized Common Knowledge

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Outline

- Common Knowledge and Relativized Common Knowledge
- Syntax of the Language
- Semantics of the Language
- Annotated Formulæ
- A Sequent System
- Advantages and Disadvantages
- Future Work
- Bibliography

Common Knowledge:
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Relativized Common Knowledge:
Every path consisting of worlds where ψ holds, ends in a path where φ holds. This may be expressed with a *release* formula.

Syntax of the Language

We have a set of atomic propositions Π whose elements are denoted by p, q, and a finite set of agents $A = \{1, \ldots, n\}$.

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathsf{AK}_G\varphi \mid \mathsf{EK}_G\varphi \mid \varphi\mathsf{R}_G\varphi \mid \varphi\mathsf{U}_G\varphi$$

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Negation rules:

$$\neg\neg p = p \quad \neg(\varphi \vee \psi) = \neg\varphi \wedge \neg\psi \quad \neg(\varphi \wedge \psi) = \neg\varphi \vee \neg\psi$$
$$\neg\mathsf{AK}_G\varphi = \mathsf{EK}_G\neg\varphi \qquad \neg\mathsf{EK}_G\varphi = \mathsf{AK}_G\neg\varphi$$
$$\neg\varphi\mathsf{R}_G\psi = \neg\varphi\mathsf{U}_G\neg\psi \quad \neg\varphi\mathsf{U}_G\psi = \neg\varphi\mathsf{R}_G\neg\psi$$

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Besides:

- *Sequents* are finite sets of formulæ.
- *Annotations* are finite sets of sequents.

Semantics of the Language

Definition (Kripke Structures)

A *Kripke Structure* for a set of agents $A = \{1, \ldots, n\}$ and a set of atomic propositions Π is a triple $\mathfrak{M} = (S, \mathcal{R}, v)$ where:

- S is a set of states.
- $\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ where is a set of n reflexive binary relations on S (one for each agent.)
- $v : \Pi \mapsto 2^S$.

Semantics of the Language

Definition (Semantics of the Language)

Given a Kripke Structure $\mathfrak{M} = (S, \mathcal{R}, v)$ and a state $s_0 \in S$, the satisfiability relation $\models$ is defined as follows:

- $(\mathfrak{M}, s_0) \models p$ iff $s_0 \in v(p)$.
- $(\mathfrak{M}, s_0) \models \neg\varphi$ iff $(\mathfrak{M}, s_0) \not\models \varphi$
- $(\mathfrak{M}, s_0) \models \varphi \vee \psi$ ($\varphi \wedge \psi$) iff $(\mathfrak{M}, s_0) \models \varphi$ or (and) $(\mathfrak{M}, s_0) \models \psi$

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- $(\mathfrak{M}, s_0) \models \varphi \vee \psi$ $(\varphi \wedge \psi)$ iff $(\mathfrak{M}, s_0) \models \varphi$ or (and) $(\mathfrak{M}, s_0) \models \psi$
- $(\mathfrak{M}, s_0) \models \mathsf{AK}_G\varphi$ $(\mathsf{EK}_G\varphi)$ iff for all states (some state) s_1 such that $(s_0, s_1) \in \mathcal{R}_i$, $i \in G$, $(\mathfrak{M}, s_1) \models \varphi$
- $(\mathfrak{M}, s_0) \models \varphi \mathsf{R}_G \psi$ iff for all G-paths $s_0, s_1, \ldots$ either there is a state s_m such that $(\mathfrak{M}, s_m) \models \varphi$ and $(\mathfrak{M}, s_j) \models \psi$ for all $j \leq m$, or $(\mathfrak{M}, s_j) \models \psi$ for all j if no such state exists.
- $(\mathfrak{M}, s_0) \models \varphi \mathsf{U}_G \psi$ iff there is some G-path $s_0, \ldots, s_m$ such that $(\mathfrak{M}, s_m) \models \psi$ and $(\mathfrak{M}, s_j) \models \varphi$ for all $j < m$.

Annotated Formulæ

An annotated formula has the form

$$\varphi \mathsf{R}_{G[H]} \psi$$

where $H = \Gamma_1, \ldots, \Gamma_m$ is an annotation.

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Corresponding formulæ:

- The corresponding formula of a sequent $\Gamma = \varphi_1, \ldots, \varphi_p$ is $\Gamma' = \varphi_1 \vee \ldots \vee \varphi_p$.
- The corresponding formula of an annotation $H = \Gamma_1, \ldots, \Gamma_m$ is $H' = \Gamma'_1 \wedge \ldots \wedge \Gamma'_m$
- The corresponding formula of an annotated formula $\varphi \mathsf{R}_{G[H]} \psi$ is $(\neg H' \vee \varphi) \mathsf{R}_G (\neg H' \vee \psi)$

A Sequent System, First Attempt

$$\text{id}\ \frac{}{\Gamma, p, \neg p} \qquad \text{ref}\ \frac{}{\Gamma, \mathsf{EK}_G\varphi, \neg\varphi} \qquad \vee\,\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi} \qquad \wedge\,\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

$$\mathsf{U}\ \frac{\Gamma, \varphi, \psi \qquad \Gamma, \mathsf{EK}_G(\varphi \mathsf{U}_G \psi), \psi}{\Gamma, \varphi \mathsf{U} \psi} \qquad \mathsf{R}\ \frac{\Gamma, \psi \quad \Gamma, \mathsf{AK}_G(\varphi \mathsf{R}_G \psi), \varphi}{\Gamma, \varphi \mathsf{R}_G \psi}$$

$$\mathsf{K}\ \frac{\alpha_i, \Gamma}{\mathsf{AK}_{G_1}\alpha_1, \mathsf{EK}_{G_1}\Gamma, \ldots, \mathsf{AK}_{G_m}\alpha_m, \mathsf{EK}_{G_m}\Gamma, p_1, \ldots, p_q}$$

Intermezzo: “Good” and “Bad” Repeats

$$(\top \ \mathsf{U}_G \neg\varphi), (\bot \ \mathsf{R}_G \varphi)$$

Intermezzo: "Good" and "Bad" Repeats

$$\mathsf{R}\ \frac{\varphi, (\top \mathsf{U}_G \neg\varphi) \qquad\qquad (\top \mathsf{U}_G \neg\varphi), \mathsf{AK}_G(\bot \mathsf{R}_G \varphi)}{(\top \mathsf{U}_G \neg\varphi), (\bot \mathsf{R}_G \varphi)}$$

Intermezzo: "Good" and "Bad" Repeats

$$
\mathsf{R}\,\dfrac{\mathsf{U}\,\dfrac{\varphi, \neg\varphi, \mathsf{EK}_G(\top\ \mathsf{U}_G\varphi)}{\varphi, (\top\ \mathsf{U}_G\neg\varphi)} \qquad \mathsf{U}\,\dfrac{\neg\varphi, \mathsf{EK}_G(\top\ \mathsf{U}_G\neg\varphi), \mathsf{AK}_G(\bot\ \mathsf{R}_G\varphi)}{(\top\ \mathsf{U}_G\neg\varphi), \mathsf{AK}_G(\bot\ \mathsf{R}_G\varphi)}}{(\top\ \mathsf{U}_G\neg\varphi), (\bot\ \mathsf{R}_G\varphi)}
$$

(OK)

$$\dfrac{\dfrac{\varphi, \neg\varphi, \mathsf{EK}_G(\top \mathsf{U}_G \varphi)}{\varphi, (\top \mathsf{U}_G \neg\varphi)}\,\mathsf{U} \qquad \dfrac{\neg\varphi, \mathsf{EK}_G(\top \mathsf{U}_G \neg\varphi), \mathsf{AK}_G(\bot \mathsf{R}_G \varphi)}{(\top \mathsf{U}_G \neg\varphi), \mathsf{AK}_G(\bot \mathsf{R}_G \varphi)}\,\mathsf{U}}{(\top \mathsf{U}_G \neg\varphi), (\bot \mathsf{R}_G \varphi)}\,\mathsf{R}$$

$$\mathsf{R}\;\dfrac{\mathsf{U}\;\dfrac{\overset{\textbf{(OK)}}{\varphi, \neg\varphi, \mathsf{EK}_G(\top\ \mathsf{U}_G\varphi)}}{\varphi, (\top\ \mathsf{U}_G\neg\varphi)} \qquad \mathsf{U}\;\dfrac{\mathsf{K}\;\dfrac{(\top\ \mathsf{U}_G\neg\varphi), (\bot\ \mathsf{R}_G\varphi)}{\neg\varphi, \mathsf{EK}_G(\top\ \mathsf{U}_G\neg\varphi), \mathsf{AK}_G(\bot\ \mathsf{R}_G\varphi)}}{(\top\ \mathsf{U}_G\neg\varphi), \mathsf{AK}_G(\bot\ \mathsf{R}_G\varphi)}}{(\top\ \mathsf{U}_G\neg\varphi), (\bot\ \mathsf{R}_G\varphi)}$$

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$$\mathsf{R}\;\dfrac{\mathsf{U}\;\dfrac{\begin{array}{c}\textbf{(OK)}\\ \varphi,\neg\varphi,\mathsf{EK}_G(\top\;\mathsf{U}_G\varphi)\end{array}}{\varphi,(\top\;\mathsf{U}_G\neg\varphi)}\qquad \mathsf{U}\;\dfrac{\mathsf{K}\;\dfrac{\begin{array}{c}\textbf{(OK)}\\ (\top\;\mathsf{U}_G\neg\varphi),(\bot\;\mathsf{R}_G\varphi)\end{array}}{\neg\varphi,\mathsf{EK}_G(\top\;\mathsf{U}_G\neg\varphi),\mathsf{AK}_G(\bot\;\mathsf{R}_G\varphi)}}{(\top\;\mathsf{U}_G\neg\varphi),\mathsf{AK}_G(\bot\;\mathsf{R}_G\varphi)}}{(\top\;\mathsf{U}_G\neg\varphi),(\bot\;\mathsf{R}_G\varphi)}$$

The Sequent System Revisited

$$\text{id}\ \frac{}{\Gamma, p, \neg p} \qquad \text{ref}\ \frac{}{\Gamma, \mathsf{EK}_G\varphi, \neg\varphi} \qquad \vee\ \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi} \qquad \wedge\ \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

$$\mathsf{U}\ \frac{\Gamma, \varphi, \psi \quad \Gamma, \mathsf{EK}_G(\varphi \mathsf{U}_G \psi), \psi}{\Gamma, \varphi \mathsf{U} \psi} \qquad \mathsf{R}\ \frac{\Gamma, \psi \quad \Gamma, \mathsf{AK}_G(\varphi \mathsf{R}_G \psi), \varphi}{\Gamma, \varphi \mathsf{R}_G \psi}$$

$$\mathsf{K}\ \frac{\alpha_i, \Gamma}{\mathsf{AK}_{G_1}\alpha_1, \mathsf{EK}_{G_1}\Gamma, \ldots, \mathsf{AK}_{G_m}\alpha_m, \mathsf{EK}_{G_m}\Gamma, p_1, \ldots, p_q}$$

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$$\text{rep}\ \frac{}{\Gamma, \varphi \mathsf{R}_{G[H,\Gamma]}\psi} \qquad \text{foc}\ \frac{\Gamma, \varphi \mathsf{R}_{G[\emptyset]}\psi}{\Gamma, \varphi \mathsf{R}_G \psi}$$

$$\mathsf{R}_H\ \frac{\Gamma, \psi \qquad \Gamma, \mathsf{AK}_G(\varphi \mathsf{R}_{G[H,\Gamma]}\psi), \varphi}{\Gamma, \varphi \mathsf{R}_{G[H]}\psi}$$

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– No syntactic cut-elimination

Future Work

- Look further on the problem of syntactic cut elimination

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- To look for better formulations of the calculus

Bibliography

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