Relative Models of Constructive Set Theory

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Relative Models of Set Theory

- Use set theory in the background and in the foreground
- Consider models which are classes
- Examples
  - Inner Models
  - Forcing
  - Realizability Models (constructive settings only)

Heyting Models

- Intuitionistic Analogon to Forcing with Boolean Algebras
- Input: complete Heyting Algebra H
- Output: Class Model of constructive Set Theory
- (In predicative settings, formal topologies are preferable to Heyting algebras)

The Universe

- Sets contain not only information about what elements lie in them, but also about the truth value of an element lying in them
- All sets are hereditarily H-valued
- Ranks of V(H) defined recursively by

$$V(H)_\alpha = \bigcup_{\beta \in \alpha} \left\{ f : a \to H \,\middle|\, a \subseteq V(H)_\beta \right\}$$

Semantics

- Assign truth values to formulas with parameters from V(H) recursively, e.g.

$$[\![\Phi_1 \wedge \Phi_2]\!] = [\![\Phi_1]\!] \wedge [\![\Phi_2]\!]$$

- (For atomic formuluae, a simultanious recursion over the parameters in necessary for extensionality)
- Formulas with value T are said to hold in the model

Realizability Models

- Originally known from arithmetic
- Arise from BHK-interpretation of intuitionistic logic
- Input: Partial Combinatory Algebra
- Output: Class Model of constructive Set Theory

The Universe

- Sets contain not only information about what elements lie in them, but can also give a computational content for them (realizer)
- V(A) defined recursively

$$V(A)_\alpha = \bigcup_{\beta \in \alpha} \wp\left(A \times V(A)_\beta\right)$$

Semantics

- Define recursively on Φ a realizability relation $e \Vdash \Phi$
- (For atomic formuluae, a simultanious recursion over the parameters in necessary for extensionality)
- A formula realized by any realizer is said to hold in the model

A common generalization

- Upgrade formal topology with application operation
- Get rid of the equivalence relation $t \simeq s$
- Use instead a partial order $t \trianglelefteq s$
- Idea: Some information / credibility may be lost by application

=> applicative Topologies

Applicative Topologies

- Formal Topology S with partial binary operation and elements k, s
- Write t $\trianglelefteq$ t' if one term denotes exactly when the other one does and when in this case the value of t is covered by the singleton of the value of t'.
- Specify some realizers „convincing“, in particular k und s

Axioms

1. $pp' \downarrow, a \lhd p, b \lhd p' \to ab \lhd pp'$
2. $xy \downarrow, x \in \nabla, y \in \nabla \to xy \in \nabla$
3. $kxy \unlhd x$
4. $sxy \downarrow$
5. $sxyz \unlhd xz(yz)$
6. $\nabla \ni x \lhd \emptyset \to \perp$

The Universe

- Again, sets are hereditarily valued by elements of the input (i.e. the applicative topology)
- So V(S) is defined recursively by

$$V(S)_\alpha = \bigcup_{\beta\in\alpha} a \in \wp\left(S \times V(S)_\beta\right)$$

- (As a technical nicety, let all sets be saturated with respect to covering)

Semantics

1. $e \Vdash \bot$ *falls* $e \lhd \emptyset$

2. $e \Vdash x \dot{\in} y$ *falls* $e \lhd y^{-1}x$

3. $e \Vdash x \in y$ *falls* $e \lhd \{f \in S | \exists z \in Bi(y).\ lf \Vdash z \dot{\in} y \wedge rf \Vdash x = y\}$

4. $e \Vdash x = y$ *falls* $\forall z \in Bi(x)\ \forall f \Vdash z \dot{\in} x\ lef \Vdash z \in y$ *und*
   $\forall z \in Bi(y)\ \forall f \Vdash z \dot{\in} y\ ref \Vdash z \in x$

5. $e \Vdash \phi \wedge \psi$ *falls* $le \Vdash \phi \wedge re \Vdash \psi$

6. $e \Vdash \phi \vee \psi$ *falls* $e \lhd \{f \in S | (lf \unlhd l \wedge rf \Vdash \phi) \vee (lf \unlhd r \wedge rf \Vdash \psi)\}$

7. $e \Vdash \phi \rightarrow \psi$ *falls* $\forall f \in S.\ f \Vdash \phi \rightarrow ef \Vdash \psi$

8. $e \Vdash \forall x \phi(x)$ *falls* $\forall a \in V(S) e \Vdash \phi[a]$

9. $e \Vdash \exists x \phi(x)$ *falls* $e \lhd \{f \in S | \exists a \in V(S)\ f \Vdash \phi[a]\}$

Advantages

- Common generalization of realizability and Heyting models
- Higher abstraction level leads to increased efficiency in proofs, more general results (partly also for the special cases)
- Some interesting applicative topologies are really new, lead to new results

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  (so the construction works also for IZF)

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- Choice principles only to a limited extent
- Regular Extension Axiom (REA)
- Relation Reflection Scheme (RRS)
- Further extension axioms (*REA, *2-REA)

Example:
Relation Reflection Scheme

- DC: $a \in G \wedge D : G \rightrightarrows G$

  $\Rightarrow \exists f : \mathbb{N} \to G.\, f(0) = a \wedge D(f(n), f(n+1))$

- RDC: $a \in \Gamma \wedge \Delta : \Gamma \rightrightarrows \Gamma$

  $\Rightarrow \exists f : \mathbb{N} \to \Gamma.\, f(0) = a \wedge \Delta(f(n), f(n+1))$

- RRS – what takes DC to RDC:

  $a \in \Gamma \wedge \Delta : \Gamma \rightrightarrows \Gamma \Rightarrow \exists G \subseteq \Gamma.\, a \in G \wedge \Delta : G \rightrightarrows G$

Realizing RRS

- Realizer of RSS must take realizers f of $a \in \Gamma \wedge \Delta : \Gamma \rightrightarrows \Gamma$ to realizers of $\exists G \subseteq \Gamma .\, a \in G \wedge \Delta : G \rightrightarrows G$
- Consider $\Gamma_{int} := \{(e,g) \mid e \Vdash \Gamma(g)\}$
  $(f_0, a) \in \Gamma_{int}$
- For all $(e,g) \in \Gamma_{int}$ exist $(f_1 e, g') \in \Gamma_{int}$ such that $f_2 \Vdash \Delta(g,g')$
- Gives rise to binary relation Δ_{int} on Γ_{int}

Realizing RRS (continued)

- By RRS in the background theory, find $G \subseteq \Gamma_{int}$ with $(f_0, a) \in G$ and:
- For all $(e,g) \in G$ exist $(f_1 e,g') \in G$ such that $f_2 \Vdash \Delta(g,g')$
- So a realizer which can be easily obtained from f realizes

$$\exists G \subseteq \Gamma.\, a \in G \wedge \Delta : G \rightrightarrows G$$

The End

- Questions
- Comments
- ...