Tutorial at LC '08. Compact spaces and definability

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Introduction

- The aim is to discuss various relationships between definable sets and compact (always Hausdorff) spaces in first order logic, touching on current research in model theory. The lectures are directed towards the general logic audience, including (descriptive) set theorists and those in "algebraic logic", but should be of interest to model-theorists too. But I will say less than promised in the abstract.
- Prerequisites: familiarity with basics of first order logic and model theory such as formula, structure, complete theory, compactness theorem, Lowenheim-Skolem theorem, saturated models, as well as basic topology.
- What is called *algebraic logic* is, I believe, about those algebraic and topological structures intrinsically associated to logic or logics (but correct me if I am wrong). These lectures are in a sense about algebraic logic, but within and at the boundaries of the first order framework.

Type spaces I

- Let T be a complete first order theory in language L, say 1-sorted for simplicity and without finite models. What (locally) compact spaces, groups,.., can be intrinsically attached to T (as invariants of T), and in what sense can they be viewed as definable sets?
- A bad answer: Let $T = RCF = Th(\mathbb{R}, +, \cdot, <)$. Then $\mathbb{R}$ is the unique locally compact model (with topology induced by the ordering). But in what sense can this model, and its topology, be recovered from T, or characterized purely model-theoretically? It is not prime, or saturated... In fact there is a positive answer and we will come back to it.
- Of course there are some compact spaces intrinsically attached to T. Namely the type spaces.

Type spaces II

- For each n let $F_n(T)$ be the Lindenbaum algebra of T, namely the Boolean algebra of L-formulas $\phi(x_1, .., x_n)$ (in free variables $x_1, .., x_n$)), up to equivalence moduli T.
- The space $S_n(T)$ is the Stone space of $F_n(T)$, namely the set of ultrafilters on $F_n(T)$, or complete n-types of T. The basic open sets of $S_n(T)$ are of the form $\{p(\bar{x}) : \phi(\bar{x}) \in p\}$, for $\phi(\bar{x})$ a formula. As such $S_n(T)$ is totally disconnected, and possibly not so interesting from the point of view of geometry.
- The theory T can actually be presented as a "type-space functor", namely the functor which takes a natural number n to $S_n(T)$. (What are the morphisms?)
- Beware: $S_2(T)$ is NOT $S_1(T) \times S_1(T)$, even as a set. It is because of this that model theory is not reduced to topology.
- This is the same phenomenon as with the Zariski topology on an algebraic variety X. The Zariski topology on $X \times X$ is not the product of the Zariski topology on X with itself.

Type spaces III

- Likewise, if $M \models T$ and $A \subseteq M$, we have $S_n(Th((M,a)_{a\in A})$ which we often write as $S_n(A)$ and call the set of complete n-types over A. (But remember this depends on $Th((M,a)_{a\in A})$.)
- So $S_n(A)$ is the Stone space of the natural Boolean algebra $F_n(A)$.
- If $M \models T$ and $A \subseteq M$ and $b \in M^n$, we have $tp(b/A) \in S_n(A)$. Moreover for any $p \in S_n(A)$ there is an elementary extension of M in which p is "realized".

Category of definable sets I

- Naively, a definable set is $\phi(M) = \{a \in M^n : M \models \phi(a)\}$, for some model M of T, and formula $\phi(x_1, .., x_n)$ of L.
- It is more reasonable to define a definable set as a functor F_ϕ from $Mod(T)$ (with elementary embeddings as morphisms) to $Sets$ determined by some formula $\phi(x_1, .., x_n)$: namely $F_\phi(M) = \phi(M)$.
- As such the set of definable sets (of n-tuples) identifies with $F_n(T)$.
- Sometimes we write a definable set as X or X_ϕ and talk about $X(M)$ for a model M.
- Likewise we can speak of sets definable with parameters, or A-definable sets, or sets defined over A.

Category of definable sets II

- Can a definable set be viewed naturally as a compact space?
- Well consider the formula $0 \leq x \leq 1$ in RCF, then in the model $\mathbb{R}$ it defines the unit interval $[0,1](\mathbb{R})$ (a compact space). But this does not count, as remarked earlier.
- If $\phi(M)$ is finite for some $M \models T$ then $\phi(M)$ has the same finite size for all M, and the functor F_ϕ has constant value a fixed finite set, which of course IS a compact space (with discrete topology).
- But once $\phi(M)$ is infinite for some model M then by the compactness theorem, $|\phi(N)|$ is unbounded, as N varies, and there is no sense in which the formula, definable set, or functor can be viewed as a compact object.
- Similarly for "type-definable" or "$\wedge$-definable" sets. A type-definable set is given by a collection $\Phi(x_1,..,x_n)$ of formulas, where for a model M, $\Phi(M) = \{a \in M^n : M \models \phi(\bar{x}) \text{ for all } \phi \in \Phi\}$.

Category of definable sets III

- Then if $\Phi(M)$ is infinite for some model M then by compactness $|\Phi(N)|$ is unbounded as N varies over models of T.
- And if $\Phi(M)$ is finite in all models M of T then Φ is equivalent to a single formula ϕ with F_ϕ constant valued.
- We can slightly enlarge our notion of definability by considering quotients by definable equivalence relations. That is let X be a definable set (even type-definable set), and E a definable equivalence relation on X (meaning what?). Then $(X/E)(M) =_{def} X(M)/E(M)$ for any model M.
- Then EITHER $|(X/E)(M)|$ is unbounded as M varies, OR $(X/E)(M)$ has constant value which is a finite set.
- So far the only definable sets which have a chance of being considered compact sets are the finite ones. But a slight twist will produce something new.

Hyperdefinable sets I

- Let X be a definable (or even type-definable) set and let now E be a *type-definable* equivalence relation on X (meaning what??). (Where X, E could be defined with parameters from some model.)
- For any model M over which X, E are defined, define $(X/E)(M)$ to be $X(M)/E(M)$. We call (the functor) X/E a hyperdefinable set.

Example 1.1

The type space $S_n(T)$ "is" a hyperdefinable set (defined with no parameters).

Proof.

Hyperdefinable sets II

- Consider the case $n = 1$. Let $E(x, y)$ be the type-definable equivalence relation given by $\{\phi(x) \leftrightarrow \phi(y) : \phi(x) \in L\}$, and X be defined by $x = x$. Then as long as all 1-types are realized in M we have a tautological bijection between $S_1(T)$ and $(X/E)(M)$.
- So in fact X/E is eventually constant, that is if N is an elementary extension of M and all 1-types are realized in M then $(X/E)(M) = (X/E)(N)$.
- We can even recover (tautologically) the topology on $S_1(T)$: A subset C of X/E is closed if there is a partial type $\Sigma(x)$ without parameters such that $\pi^{-1}(C)(M) = \Sigma(M)$ for any model M. More on this to be said later.

$\square$

Hyperdefinable sets III

Definition 1.2

Let X/E be a hyperdefinable set. Call X/E bounded if it is eventually constant. Namely there is a model M_0 (over which X, E are defined) such that whenever $M_0 \prec M$ then $(X/E)(M) = (X/E)(M_0)$. Equivalently, for $\bar{M}$ a sufficiently saturated model, $|(X/E)(\bar{M})| < |\bar{M}|$.

- Some remarks.
- So through Example 1.1 we have examples of bounded hyperdefinable sets which do not reduce to finite sets, even though there X/E is "profinite".
- Definition 1.1 seems very semantic but in fact the notion of "bounded hyperdefinable set" can be obtained purely syntactically, working with $Th(M_0, m)_{m \in M_0}$ where M_0 is a model over which the data are defined.

Hyperdefinable sets IV

- It is convenient at this point to replace the category $Mod(T)$ by a fixed very saturated model $\bar{M}$ of T (whose existence may depend on set theory). So for some inaccessible cardinal $\bar{\kappa} > |T|$, $\bar{M}$ is $\bar{\kappa}$ saturated and of cardinality $\bar{\kappa}$.
- $\bar{M}$ is a kind of "proper class" or universe. "Small" or "bounded" means of cardinality $< \kappa$. $M, N, ..$ denote small elementary substructures of $\bar{M}$, and $A, B, ..$ small subsets of $\bar{M}$. Partial types $\Phi(\bar{x})$ are meant to be over small sets of parameters.
- Identify a definable, or type-definable, or hyperdefinable, set X with $X(\bar{M})$.
- Then it is a fact/theorem that a hyperdefinable set X is bounded just if $X(\bar{M})$ is bounded, i.e. of cardinality $< \bar{\kappa}$.
- This may offend certain sensibilities, but again everything I say will have an equivalent syntactic presentation.

Hyperdefinable sets V

Theorem 1.3

Let X/E be a bounded hyperdefinable set, with X, E defined over a model M_0. Let $\pi : X \to X/E$ be the canonical projection. Define $C \subseteq X/E$ to be closed if
(*) $\pi^{-1}(C) \subseteq X$ is type-definable (with parameters).
Then
(i) this equips X/E with a compact (Hausdorff) topology, which we call the logic topology. Moreover the topology is the same if in (*) we only require type-definability with parameters from M_0.
(ii) In particular, for $b \in X$ $(= X(\bar{M}))$, $\pi(b)$ depends only on $tp(b/M_0)$. Hence $\pi : X \to X/E$ factors through the relevant type space $S_n(M_0)$, and in fact X/E with its logic topology, is a quotient of the space $S_n(M_0)$.

Proof.

Exercise, using the compactness theorem □

Standard part maps I

- So we have seen that bounded hyperdefinable sets are, as compact spaces, continuous images of type spaces.
- Are there interesting, in particular non totally disconnected, spaces arising this way?
- We return to the example RCF mentioned at the beginnibg of the talk.
- So take T to be RCF and I the definable set $0 \leq x \leq 1$ (defined without parameters). That is I is the interpretation of this formula in the model $\bar{M}$ (a saturated real closed field containing $\mathbb{R}$ as an elementary substructure).
- $I(\mathbb{R})$ is the standard unit interval, so I can be viewed as the "nonstandard" $I(\mathbb{R})$.
- From nonstandard analysis we have the usual standard part map $\pi : I \to I(\mathbb{R})$.

Standard part maps II

- So for $x, y \in I$, $\pi(x) = \pi(y)$ iff $|x - y| < 1/n$ for each positive natural number n. (Where $|x - y|$ means in the sense of the real closed field $\bar{M}$).
- Thus the equivalence relation $\pi(x) = \pi(y)$ is type-definable with no parameters, and we call it E. So I/E is a bounded hyperdefinable set which identifies set-theoretically with $I(\mathbb{R})$.

Theorem 2.1

The logic topology on I/E coincides with the usual (Euclidean, or order) topology on $I(\mathbb{R})$.

Proof.

Exercise. But note there is something interesting going on here as there is no explicit mention of $<$ in the definition of the logic topology. Note also that E is NOT a conjunction of definable equivalence relations. $\square$

Standard part maps III

- Is there something canonical about the type-definable equivalence relation E (being infinitesimally close)?
- Well, after identifying 0 and 1, addition modulo 1 equips I with a group structure, and E is essentially the finest bounded type-definable (even with parameters) equivalence relation which is invariant under the group law.
- So E is something canonical, and by Theorem 1.4 we have recovered from the theory RCF the real unit interval as a compact topological space, without ever imposing from outside any topologies, and also giving a good answer to our question at the beginning.

Domination I

- We have Lebesgue measure on $I(\mathbb{R})$, which coincides with Haar measure on the real Lie group S_1 (after identifying 0 and 1). Call it h. It is not hard, using o-minimality of RCF to see that h has a *unique* extension to a finitely additive probability measure on the definable subsets of I in $\bar{M}$ (i.e. to a so-called global Keisler measure on I).
- Namely every definable subset of I in $\bar{M}$ is a finite union of intervals and points, and we are forced to assign 0 to "infinitesimal" intervals.
- We will extend this to the n-dimensional unit cube I^n.

Domination II

Lemma 2.2

Let $\pi : I^n \to I^n(\mathbb{R})$ be the standard part map. Let $X \subseteq I^n$ be definable (with parameters). Let $C \subseteq I^n(\mathbb{R})$ be the set of those c such that the fibre $\pi^{-1}(c)$ intersects both X and the complement of X. THEN C has Haar measure 0.

- Lemma 2.2 says that the definable set I^n is dominated by the compact set $(I^n(\mathbb{R}), h)$ equipped with its measure h, via the map π.
- This is essentially a special case of results on standard part maps proved a few years ago by Berarducci and Otero.

Domination III

Proof.

- The first ingredient is "definability of types over $\mathbb{R}$".
- What this says is that if $X \subseteq \bar{M}^n$ is definable with parameters. Then $X \cap \mathbb{R}^n$ is definable (with parameters from $\mathbb{R}$ of course) in the structure $(\mathbb{R}, +, \cdot, <)$.
- By the way, $\mathbb{R}$ is the unique model of RCF with the property that all types over it are definable. So this property recovers $\mathbb{R}$ from RCF at least as a structure (rather than topological space).
- The first step is to prove that the set C in the statement of Lemma 2.2 is definable in the structure $\mathbb{R}$.
- Let's do this. Let $Z = \{(b, \epsilon) \in \bar{M}^{n+1} : b \in I^n, \epsilon > 0$ and the ϵ ball around b contains points in X and points in $X^c\}$.
- So Z is definable in $\bar{M}$ over the same parameters used to define X.

Domination IV

- By the definability of types fact mentioned above, $W = Z \cap \mathbb{R}^n$ is definable (with parameters) in the structure $(\mathbb{R}, +, \cdot, <)$.
- Then one can check that $C = \{b \in I^n(\mathbb{R})\colon$ for all $\epsilon > 0$ in $\mathbb{R}$, $(b, \epsilon) \in W\}$, clearly definable in $\mathbb{R}$.
- In general (measurable) subsets of $I^n(\mathbb{R})$ with positive measure need not have interior. But for sets definable in $(\mathbb{R}, +, \cdot, <)$ they must (via o-minimal dimension etc.).
- So assuming, for a contradiction, that $h(C) > 0$, C must have interior in $\mathbb{R}^n$, hence contains a definable (in $\mathbb{R}$) open set U.
- Now consider $U(\bar{M})$, namely the interpretation of the formula defining U in the big model. This is an open subset of I^n defined over $\mathbb{R}$.
- From o-minimality we can deduce quite easily that either $X \cap U(\bar{M})$ or $X^c \cap U(\bar{M})$ contains an open $\mathbb{R}$-definable set.

- This is a contradiction (why?) and the lemma is proved.
- The same proof works with RCF replaced by $Th(\mathbb{Q}_p, +, \cdot)$ and I replaced by the valuation ring (as the two main ingredients, definability of types over the "standard" model, and cell decomposition for definable sets, remain valid here).

Corollary 2.3

Lebesgue measure on $I^n(\mathbb{R})$ *lifts to a* unique *(global) Keisler measure on* I^n.

Domination VI

- A Keisler measure (over A) on a definable set X is a finitely additive probability measure on the definable (over A) subsets of X.
- Lemma 2.2 and Corollary 2.3 extend easily to compact Lie groups definable in $\mathbb{R}$ equipped with their Haar measure.
- Results by Koiran and Karpinski-Macintyre on the "approximate definability of Lebesgue measure" on the real n-dimensional unit cube, in the structure $(\mathbb{R}, +, \cdot, <)$, follow quickly from Corollary 2.3.

- We point out a recent rather serious generalization of the "domination" results of the last section.
- One works now with an arbitrary o-minimal expansion T of RCF. (In fact given other recent results we are close to being able to work with an arbitrary o-minimal theory.)
- Again $\bar{M}$ is a saturated model. We take G to be a "definably compact" definable group, where definably compact can be thought of as living as a closed and bounded definable set in some $\bar{M}^n$.
- So I^n (considered as a group) is replaced by G.
- The new situation differs from RCF in at least two ways.
- First, T may no longer have a model whose underlying ordered set (or field) is $\mathbb{R}$. So there is no extrinsic standard part map entering the picture.

- Secondly, even if there were, G may no longer be defined over $\mathbb{R}$, or even "descend" to $\mathbb{R}$ (that is be definably isomorphic to a group defined over $\mathbb{R}$).
- Example: $G = A(\bar{M})$, A abelian variety over $\bar{M}$ with trivial $\mathbb{R}$-trace.

Theorem 2.4

Let G be a definably compact, definably connected, definable group in $\bar{M} \models T$. (T o-minimal as above.) Then G has a unique smallest type-definable subgroup of bounded index G^{00}. Equipped with the logic topology, G/G^{00} is a compact Lie group with dimension equal to the o-minimal dimension of G. Moreover G is dominated by $(G/G^{00}, h)$ under the canonical $\pi : G \to G/G^{00}$, where h is Haar measure on G/G^{00}.

G^{00} III

- I will discuss some of the words and notions in the statement of Theorem 2.4.
- For T an arbitrary theory, G a definable group and A a (small) set of parameters over which G is defined, let G^0_A be the intersection (conjunction) of all A-definable subgroups of G of finite index, and let G^{00}_A be the smallest type-definable over A subgroup of G of bounded index.
- Then $G^{00}_A \subseteq G^0_A$.
- If G^0_A does not depend on the choice of A, we call it G^0, the definably connected component of G. Likewise for G^{00} which we may call the type-definably connected component of G.
- Under the logic topology, G/G^{00} is a compact topological group, and G/G^0 is its maximal profinite quotient.
- If T is ω-stable (countably many types over any countable model), then $G^0 = G^{00}$ and is definable with finite index.

G^{00} IV

- If T is stable (at most λ^ω many types over any model of size λ) then $G^0 = G^{00}$ also, but G^{00} may be an infinite intersection of definable subgroups of finite index.
- For T without the independence property, G^{00} exists but may be strictly smaller than G^0.
- For example if T is o-minimal, then G^0 is definable of finite index (G is definably connected by finite). So as we have seen G^{00} may not equal G^0.
- A Lie group is a real analytic manifold with real analytic group structure. When we say G is a compact Lie group we mean it is the underlying topological group of a compact Lie group.
- Arbitrary compact groups are obtained from finite (discrete) groups and connected compact Lie groups by taking inverse (or projective) limits.

Proof.

(of Theorem 2.4)

- The first part (G and G/G^{00} have same dimension) was the topic of Peterzil's tutorial last year.
- The moreover part is more recent.
- The crucial case is where G is commutative.
- I will not attempt even to outline the proof, but this proof yields a rather attractive picture: in the Shelah expansion $\bar{M}^*$ of $\bar{M}$ obtained by adding predicates for externally definable sets, G/G^{00} is actually definable (interpretable) and "semi-o-minimal", and moreover the topological structure it obtains this way agrees with the logic topology.

□

G^{00} VI

- A corollary of Theorem 2.4 is that G has a unique left invariant global Keisler measure (which is also its unique right invariant measure).
- In fact what I have been describing is in a sense the tip of an iceberg. In the background is an appropriate generalization of stable group theory to definable groups in theories without the independence property, and also the generalization of the machinery of forking to such theories.
- For example the uniqueness of measure statement above is the generalization of uniqueness of invariant types for connected stable groups.

KP-types and the compact Lascar group I

- The material here, although fitting in to the general scheme of the lectures, is rather old, but gives me the chance to restate a conjecture on the descriptive set-theoretic nature of a certain quotient object.
- We fix again a complete first order theory T (1-sorted), which at some point we will assume to be countable, namely in a countable language L.
- I discuss now some equivalence relations and groups intrinsically associated to T.
- Again I will work semantically (via definability) but everything has an equivalent syntactic presentation.
- Fix n and consider equivalence relations E on n-tuples from $\bar{M}$ which are type-definable without parameters, and bounded ($< |\bar{M}|$, or equivalently at most $2^{|T|}$ classes).

KP-types and the compact Lascar group II

- So for $T = RCF$ and $n = 1$ the relation, discussed in the previous lecture, of being infinitesimally close for x, y in I and equal otherwise, is such an equivalence relation. For an arbitrary theory T the relation of having the same type over $\emptyset$ is another.
- In any case, there is clearly a finest such equivalence relation, which we call E^n_{KP}. The bounded hyperdefinable (without parameters) set $\bar{M}^n/E^n_{KP}$ is sometimes called the set of KP-strong types of n-tuples.
- $Aut(\bar{M})$ acts on this (compact) space, for each n. The group of KP-strong automorphisms of $\bar{M}$ consists by definition of those automorphisms which fix pointwise each space, and the quotient of $Aut(\bar{M})$ by the group of KP-strong automorphisms, is called the KP-group or compact Lascar group of T. It is naturally a compact topological group, and is a basic invariant of the theory T (and is $*$-hyperdefinable).

KP-types and the compact Lascar group III

- In cases such as RCF (or more generally when some model of T has all elements pointwise definable), then E^n_{KP} coincides with having the same type over $\emptyset$, for all n. In this case clearly $Aut(\bar{M})$ acts trivially on the space of KP-strong types so the KP-group is trivial, and in fact the whole of the current section is vacuous.
- On the other hand if $T = ACF_0 = Th(\mathbb{C}, +, \cdot)$, then the union of the spaces of KP-strong types, for n varying, identifies with $\mathbb{Q}^{alg}$ the field of algebraic numbers, and the KP-group of T coincides with $Gal(\mathbb{Q}^{alg}/\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$. (This uses "elimination of hyperimaginaries and imaginaries" in ACF_0.)

Example 3.1

Let M be the unit circle, equipped with the circular ordering $S(x, y, z)$ as well as binary predicates $R_n(x, y)$ for each n, which hold if $d(x, y) \leq 1/n$ (suitably normalized). Let $T = Th(M)$. Then $E^1_{KP} = \{R_n(x, y) : n = 1, 2, ...\}$ and the space of KP-strong 1-types is homeomorphic to S^1.

- There is another equivalence relation which at first sight belongs to infinitary logic, but is in fact quite central to the first order context.

Definition 3.2

(i) Fix n. Then E_L^n (which will be an equivalence relation on n-tuples from $\bar{M}$) is the transitive closure of the relation: x, y are the first two members of an infinite indiscernible sequence.
(ii) The group $Autf(\bar{M})$ of Lascar strong automorphisms of $\bar{M}$, is the subgroup of $Aut(\bar{M})$ generated by $\{Aut(\bar{M}/M) : M$ a small elementary substructure of $\bar{M}\}$.

Lascar strong types and the Lascar group II

Lemma 3.3

E_L^n has the following alternative characterizations:
(i) It is the finest bounded equivalence relation on n-tuples from $\bar{M}$ which is $Aut(\bar{M})$invariant.
(ii) It is the orbit equivalence relation under the action of $Autf(\bar{M})$ on $\bar{M}^n$.

- In connection with (ii) important technical observations are (I): if a, b begin an infinite indiscernible sequence then for some model M, $tp(a/M) = tp(b/M)$, and (II): if $tp(a/M) = tp(b/M)$ then for some c, both (a, c), and (b, c) begin infinite indiscernible sequences.
- E_L refines E_{KP}.
- It was open for some time whether E_L always equals E_{KP}.
- Ziegler found a counterexample around ten years ago, which appeared in a joint paper with Casanovas, Lascar and myself: Galois groups of first order theories, JML, 2001.

Lascar strong types and the Lascar group III

- The group $Aut(\bar{M})/Autf(\bar{M})$ which acts on the "space" of Lascar strong types, i.e. on the $\bar{M}^n/E_L^n$, for all n, is called the Lascar group of T, another basic invariant.
- The KP-group is a quotient of the Lascar group, but the kernel is a rather mysterious object, which we believe could or should be understood descriptive set-theoretically.
- It is conceptually easier to study the equivalence relations E_{KP}, E_L directly.
- So the issue is the structure of the set of E_L-classes in a given E_{KP}-class.

Borel equivalence relations I

- Let us now assume T to be countable.
- We fix an E^1_{KP}-class which we call X, and want to understand X/E_L.
- So far this just concerns pointsets. Namely X is a subset of $\bar{M}$ and $E_L|X$ a certain equivalence relation on X.
- Note that X is type-definable over any element a of X as $E_{KP}(x, a)$.
- But what kind of mathematical object is X/E_L? Our "thesis" is that it belongs to descriptive set theory.
- Newelski proved some time ago that the cardinality of X/E_L is either 1 or 2^ω.
- Let M_0 be a countable model which contains an element a_0 of X.
- Let C be the set of $tp(a/M_0)$ for $a \in X$.

Borel equivalence relations II

- So C is a closed subset of $S_1(M_0)$ (defined by $E_{KP}(x, a_0)$).
- Note that C is a (compact) Polish space (as a closed subspace of the compact separable metrizable space $S_1(M_0)$).
- We remarked earlier that if $tp(a/M_0) = tp(b/M_0)$ then $E_L(a, b)$.
- Hence the projection $\pi : X \to X/E_L$ factors through C, so can be written as C/E for some equivalence relation E on the Polish space C.
- E is Borel, in fact K_σ (a countable union of compacts).
- Why? Because, for $p, q \in C$, $E(p, q)$ if there are realizations a of p and b of q such that $E_L(a, b)$. (Check...)
- There is a reasonably developed theory of the complexity of Borel equivalence relations on Polish spaces (Discuss?).

Borel equivalence relations III

- The conjecture, which strengthens Newelski's result mentioned above is:
- EITHER C/E (i.e. X/E_L) is *not concretely classifiable* OR it is trivial (i.e. one point).
- Concrete classifiability would mean that there is a Borel map $f : C \to D$ from C to another Polish space which realizes (or eliminates) E, namely $f(p) = f(q)$ iff $E(p, q)$.
- Among non concretely classifiable K_σ relations on Polish spaces, there is a "weakest" one E_0 and a "strongest" one E_{K_σ} (where the latter was found rather recently).
- The original Ziegler example has complexity E_{K_σ}. With Slawek Solecki we modified this to give an example (i.e. of a theory T and E_{KP}-class X) such that the Borel complexity of X/E_L is E_0.

References

- Proper references will be given if and when these notes are presented more formally.
- But in the meantime I should mention some names (in addition to myself) of the many mathematicians associated with the work and results discussed in these lectures. Apologies for any inadvertent omissions.
- Section 1: Folklore, Lascar...
- Section 2: Berarducci, Edmundo, Hrushovski, Onshuus, Otero, Peterzil.
- Section 3: Casanovas, Lascar, Newelski, Solecki, Ziegler.
- I am not sure about the references for the theory of Borel equivalence relations and recent results on K_σ equivalence relations.