Forcing axioms, supercompact cardinals, singular cardinal combinatorics

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Let $\theta \leq \kappa$ be infinite regular cardinals.

$\mathcal{D}$ is a θ-matrix on κ if

$$\mathcal{D} = \{D(i, \beta) : i < \theta, \beta < \kappa\}$$

where the entries $D(i, \beta)$ are sets such that for each β:

- $\bigcup_{i<\theta} D(i, \beta) = \beta$,

- $D(i, \beta) \subseteq D(j, \beta)$ if $i < j < \theta$.

θ is the width of $\mathcal{D}$ and κ is the height of $\mathcal{D}$.

Typically any $f : [\kappa]^2 \to \theta$ define a θ-matrix on κ

$$\mathcal{D}_f = \{D(i, \beta) : i < \theta, \beta < \kappa\}$$

setting $D(i, \beta) = \{\alpha < \beta : f(\alpha, \beta) < i)\}$.

Conversely any θ-matrix $\mathcal{D}$ on κ define a coloring $f : [\kappa]^2 \to \theta$ by

$$f(\alpha, \beta) = \text{least } i < \theta \text{ such that } \alpha \in D(i, \beta)$$

Definition 1 *Given a θ-matrix $\mathcal{D}$ on κ, the covering property* CP($\mathcal{D}$) *holds if there is A unbounded subset of κ such that $[A]^\theta$ is covered by $\mathcal{D}$.*

Where $\mathcal{E}$ is covered by $\mathcal{F}$ if for all $X \in \mathcal{E}$ there is $Y \in \mathcal{F}$ such that $X \subseteq Y$.

We focus our attention on the covering properties CP($\mathcal{D}$) for the following reasons:

- Strong forcing axioms like PFA and MM or strongly compact cardinals implies CP($\mathcal{D}$) for a wide range of matrices $\mathcal{D}$ as above.

- It can be seen that a number of problems concerning singular cardinal combinatorics can have an equivalent formulation in terms of the validity of the covering property CP($\mathcal{D}$) for the appropriate matrices $\mathcal{D}$.

A list of problems in singular cardinal combinatorics that can be treated by means of $\mathsf{CP}(\mathcal{D})$

- Various proofs that the proper forcing axiom PFA implies the singular cardinals hypothesis SCH: i.e the cardinal arithmetic $\kappa^{\mathrm{cf}(\kappa)} = \kappa^{+} + 2^{\mathrm{cf}(\kappa)}$.

- Various proofs that large cardinals and forcing axioms imply the negation of many square principles.

- A complete description of Shelah's approachability ideal $\mathcal{I}[\aleph_{\omega+1}]$ in models of Martin's maximum MM (joint work with Assaf Sharon).

- An analysis of the "saturation"-properties of models of strong forcing axioms (PFA, MM), this will be made precise in the sequel.

A sample on how the covering properties CP($\mathcal{D}$) **can be applied.**

We sketch a proof of the following theorem by Solovay.

Theorem 2 (Solovay) *Assume λ is strongly compact and $\kappa \geq \lambda$ has uncountable cofinality. Then $\kappa^{\aleph_0} = \kappa$.*

This theorem can be combined with another classical result by Silver to conclude that SCH holds above a strongly compact cardinal.

A sketch of the proof of Solovay's theorem:

We proceed by induction on a $\kappa \geq \lambda$ of uncountable cofinality to show $\kappa^{\aleph_0} = \kappa$. There are threee cases to consider:

- κ is a limit cardinal. Then $\kappa^{\aleph_0} = \sup_{\theta<\kappa} \theta^{\aleph_0} = \sup_{\theta<\kappa} \theta = \kappa$.

- $\kappa = \nu^+$ and ν has uncountable cofinality. Then we can use the Haussdorff formula $\kappa^{\aleph_0} = \nu^{\aleph_0} + \nu^+$ and the inductive assumption on $\nu^{\aleph_0}$ to conclude $\kappa^{\aleph_0} = \nu^+$.

- $\kappa = \nu^+$ and ν has countable cofinality. Here we will crucially use the fact that λ is strongly compact and $\kappa \geq \lambda$ is regular.

We must show $\kappa^{\aleph_0} = \kappa$ where $\kappa = \nu^+$ and ν has countable cofinality.

We fix an increasing sequence $\{\nu_n : n \in \omega\}$ of regular cardinals converging to ν.

By inductive assumption $\nu_n^{\aleph_0} = \nu_n$ for all n.

Fix for any $\beta < \kappa$ a surjection $\phi_\beta : \nu \to \beta$.

Set $\phi_\beta[\nu_n] = D(n, \beta)$.

$$\mathcal{D} = \{D(n, \beta) : n \in \omega, \beta < \kappa\}$$

is an $\aleph_0$-matrix on κ.

For the moment assume CP($\mathcal{D}$) holds.

Then there is A unbounded subset of κ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$, i.e.:

$$[A]^{\aleph_0} \subseteq \bigcup_{n\in\omega,\beta<\kappa} [D(n,\beta)]^{\aleph_0}$$

Now for each n, β,

$$|[D(n,\beta)]^{\aleph_0}| \leq \nu_n^{\aleph_0} = \nu_n < \nu$$

Finally:

$$\kappa^{\aleph_0} = |A^{\aleph_0}| \leq |\bigcup_{n<\omega,\beta<\kappa} [D(n,\beta)]^{\aleph_0}| \leq \kappa \times \nu = \kappa$$

So the proof is complete once we show that CP($\mathcal{D}$) holds.

Fact 1 *Assume* $\theta < \lambda \leq \kappa$ *are regular cardinals,* λ *is strongly compact and*

$$\mathcal{D} = \{D(i, \beta) : i < \theta, \beta < \kappa\}$$

is a θ*-matrix on* κ*. Then* CP($\mathcal{D}$) *holds.*

Proof: Fix a λ-complete uniform ultrafilter $\mathcal{U}$ on κ. Let:

$$A^i_\alpha = \{\beta > \alpha : \alpha \in D(i, \beta)\}$$

for each $\alpha < \kappa$ and $i < \theta$.

Now $\kappa \setminus \alpha \in \mathcal{U}$ for each α and $\kappa \setminus \alpha = \bigcup_{i<\theta} A^i_\alpha$.

So for each α there is $i_\alpha < \theta$ such that $A^{i_\alpha}_\alpha \in \mathcal{U}$.

Since $\kappa > \theta$ is regular there is $i < \theta$ such that for some fixed i:

$$A_i = \{\alpha < \kappa : i_\alpha = i\}$$

is unbounded in κ.

We show $[A_i]^\theta$ is covered by $\mathcal{D}$.

Pick $X \in [A_i]^\theta$.

Since $\mathcal{U}$ is λ-complete, X has size θ and $A^i_\alpha \in \mathcal{U}$ for all $\alpha \in X$, so:

$$A = \bigcap_{\alpha \in X} A^i_\alpha \in \mathcal{U}.$$

Thus there is $\beta \in A$.

Then $\alpha \in D(i, \beta)$ for all $\alpha \in X$ as was to be shown. $\square$

Solovay's theorem is now proved. $\square$

The previous example shows:

A

If for every singular cardinal $\nu > 2^{\aleph_0}$ of countable cofinality, there is an $\aleph_0$-matrix $\mathcal{D}$ on ν^+ such that:

- all its entries have size less than ν,
- CP($\mathcal{D}$) holds,

then the singular cardinal hypothesis holds.

B

In the case that $\mathcal{D}$ is a θ-matrix on a regular κ with $\theta < \lambda \leq \kappa$ and λ a strongly compact cardinal CP($\mathcal{D}$) holds.

All the problems on singular cardinal combinatorics listed before can be approached along the lines of the proof of Solovay's theorem.

The first step is to recognize that the relevant problem is reducible to the validity of CP($\mathcal{D}$) for the appropriate family of matrices $\mathcal{D}$.

The second step is to isolate the right conditions on $\mathcal{D}$ which guarantees that CP($\mathcal{D}$) can hold.

For the other problems listed before the complexity of the solutions provided by this approach may increase considerably for two obvious reasons:

- It is not apparent that a solution to the problem follows by the validity of CP($\mathcal{D}$) for some appropriate class of matrices $\mathcal{D}$.

- It is not clear whether CP($\mathcal{D}$) for the relevant family of matrices $\mathcal{D}$ can at all be consistent.

This leads to analyze in more detail the combinatorial properties of θ-matrices on κ or equivalently of the colorings $f : [\kappa]^2 \to \theta$.

Definition 3 *Let $\theta < \kappa$ be regular cardinals.*

$$\mathcal{D} = \{D(i, \beta) : i < \theta, \beta < \kappa\}$$

is a θ-covering matrix on κ if it is such that:

- $\beta = \bigcup_{i<\theta} D(i, \beta)$ *for all $i < \theta$,*

- $D(i, \beta) \subseteq D(j, \beta)$ *for all $i < j$.*

- *for all $\alpha < \beta < \kappa$ and all $i < \theta$ there is $j < \theta$ such that $D(i, \alpha) \subseteq D(j, \beta)$,*

$\beta_{\mathcal{D}} = \sup\{\mathrm{otp}(D(i, \beta)) : i < \theta, \beta < \kappa\}$.

Given a θ-covering matrix $\mathcal{D}$ on κ we say that:

- $\mathcal{D}$ is downward coherent if for all $\alpha < \beta < \kappa$ and all $i < \theta$ there is $j < \theta$ such that $D(i, \beta) \cap \alpha \subseteq D(j, \alpha)$,

- $\mathcal{D}$ is locally downward coherent if for all $X \in [\kappa]^\theta$ there is $\gamma < \kappa$ such that for all $\beta < \kappa$ and all $i < \theta$ there is $j < \theta$ such that $D(i, \beta) \cap \gamma \subseteq D(j, \gamma)$,

- $\mathcal{D}$ is normal if $\beta_{\mathcal{D}} < \kappa$.

Remark that a downward coherent matrix is locally downward coherent.

When CP($\mathcal{D}$) **fails and why** CP($\mathcal{D}$) **is in most cases a large cardinal assumption.**

Lemma 4 (Cummings and Schimmerling) *Assume κ is a regular cardinal. Then there is a normal, downward coherent, κ-covering matrix $\mathcal{D}$ on κ^+.*

Lemma 5 (Jensen) *Assume that $\square_\kappa$ holds. Then there is a normal, downward coherent, $\aleph_0$-covering matrix $\mathcal{D}$ on κ^+.*

Fact 2 *Assume that $\theta < \kappa$ are regular cardinals and $\mathcal{D}$ is a normal, downward coherent, θ-covering matrix $\mathcal{D}$ on κ. Then* CP($\mathcal{D}$) *fails.*

Corollary 6 (Solovay) *Assume λ is strongly compact. Then $\square_\kappa$ fails for all $\kappa \geq \lambda$.*

Proof of the fact: Assume to the contrary that A is an unbounded subset of κ and $[A]^\theta$ is covered by

$$\mathcal{D} = \{D(i,\beta) : i < \theta,\, \beta < \kappa^+\}.$$

where $\mathcal{D}$ is a normal, downward coherent θ-covering matrix on κ.

Since $\beta_{\mathcal{D}} < \kappa$, we can pick an initial segment of A of order-type larger than $\beta_{\mathcal{D}}$.

Let $\eta < \kappa^+$ be its supremum.

Now $A \cap \eta \not\subseteq D(i,\eta)$ for all $i < \theta$ since

$$\mathrm{otp}(A \cap \eta) > \beta_{\mathcal{D}} \geq \mathrm{otp}(D(i,\eta))$$

for all $i < \theta$.

Thus we can produce X subset of $A \cap \eta$ of size θ such that $X \setminus D(i,\eta)$ is non-empty for all $i < \theta$.

Since $[A]^\theta$ is covered by $\mathcal{D}$ we can find $\nu < \kappa$ and $i < \theta$ such that $X \subseteq D(i, \nu)$.

Now since $\mathcal{D}$ is downward coherent, we have that for some $j < \theta$, $D(i, \nu) \cap \eta \subseteq D(j, \eta)$.

Thus $X \subseteq D(j, \eta)$. Contradiction. $\square$

Theorem 7 *Assume* PFA, *let* $\kappa > \aleph_1$ *be a regular cardinal and* $\mathcal{D}$ *be a locally downward coherent,* $\aleph_0$*-covering matrix* $\mathcal{D}$ *on* κ*. Then* CP($\mathcal{D}$) *holds.*

Corollary 8 (Todorčević) *Assume* PFA*. Then* $\square_\kappa$ *fails for all regular uncountable* κ*.*

Recall that $\square_\kappa$ holds in L for all uncountable κ and it is known to be consistent with (for example) κ being measurable.

PFA **and** SCH

We can also hint how to prove that PFA implies SCH,

Lemma 9 *For all singular cardinal κ of countable cofinality, there is a normal, locally downward coherent $\aleph_0$-covering matrix $\mathcal{D}$ on κ^+.*

By PFA, CP($\mathcal{D}$) holds for any such $\aleph_0$-covering matrix $\mathcal{D}$ on κ^+.

As in the proof of Solovay's theorem, we conclude that

$\kappa^{\aleph_0} = \kappa^+$ for all κ of countable cofinality.

Rigidity of models of strong forcing axioms

MM and PFA appears to produce models of set theory in which every *"consistent"* set of size at most $\aleph_1$ *"exists"*.

How to formulate this in a suitable form?

For example in this way:

Theorem 10 (Veličković) *Assume* MM. *Let* W *be an inner model such that* $\omega_2^W = \omega_2$. *Then* $P(\omega_1) \subseteq W$.

Theorem 11 (Caicedo, Vel.) *Assume* $W \subseteq V$ *are models of* BPFA *such that* $\omega_2^W = \omega_2$. *Then* $P(\omega_1) \subseteq W$.

We want to extend these results all over the cardinals:

Conjecture 1 (Caicedo, Veličković) *Assume* $W \subseteq V$ *are models of* MM *with the same cardinals. Then* $[Ord]^{\leq\omega_1} \subseteq W$*.*

This is almost best possible, since:

- There exist $W \subseteq V$ models of MM with the same cardinals such that $[Ord]^{\omega_2} \not\subseteq W$.

- Using stationary tower forcing it is possible to produce two models of MM, $W \subseteq V$ such that $[Ord]^{\leq\omega_1} \not\subseteq W$. However the two models have different cardinals.

FIRST PROBLEM TO MATCH: FIXING THE COFINALITIES.

Results by Cummings and Schimmerling show that if $W \subseteq V$ are models of ZFC with the same cardinals and V models PFA then the two models have the same cardinals of countable cofinality.

This result can be proven using covering matrices.

Notice that if g is a generic Prikry sequence on a measurable κ, $V \subseteq V[g]$ have the same cardinals but $g \notin V$ is a countable set of ordinals.

Cummings and Schimmerling result shows that PFA cannot hold in the generic extension by Prikry forcing.

How to use covering matrices for Cummings and Schimmerling result.

Fact 3 *Assume W is an inner model and κ is a singular cardinal such that:*

- *it is regular in W,*
- $(\kappa^+)^W = \kappa^+$.

Then there is a normal, downward coherent, cf(κ)*-covering matrix $\mathcal{D}$ on κ^+.*

Proof of the fact: In W there is a normal, downward coherent, κ-covering matrix $\mathcal{D}$ on κ^+. We can use a cofinal sequence in κ of order-type cf(κ) to refine it to a normal, downward coherent, cf(κ)-covering matrix $\mathcal{D}$ on κ^+. $\square$

We mentioned that under PFA there cannot be a normal, downward coherent $\aleph_0$-covering matrix on κ^+.

Corollary 12 (Cummings and Schimmerling) *Assume κ is measurable and G is generic for Prikry forcing on κ. Then* PFA *fails in $V[G]$.*

Proof of the corollary: Prikry forcing produces two models $W \subseteq V$ with the same cardinals and such that κ has countable cofinality in V. This is incompatible with PFA. $\square$

The following theorem continues a pattern already explored by Hamkins which obtained many results of a similar vein:

Theorem 13 *Assume λ is strongly compact and $\kappa > \lambda$. Let W be an inner model such that κ is regular in W and $\kappa^{+} = (\kappa^{+})^{W}$.*
Then $\mathrm{cf}(\kappa) \geq \lambda$.

Proof of the theorem If not let $\theta = \mathrm{cf}(\kappa) < \lambda$. By the previous fact there is a normal, downward coherent θ-covering matrices $\mathcal{D}$ on κ^{+}. Since $\theta < \lambda < \kappa^{+}$, and λ is strongly compact, CP($\mathcal{D}$) holds.

This is impossible since we know that CP($\mathcal{D}$) cannot hold for any normal, downward coherent θ-covering matrices on κ^{+}. $\square$

We can also prove the analogue result for models of MM but the proof is more sophisticated and combines techniques coming from Cummings and Schimmerling with an application of the Strong Chang conjecture.

Theorem 14 *Assume* MM. *Let* κ *be singular (and strong limit). Let* W *be an inner model such that* κ *is regular in* W *and* $\kappa^+ = (\kappa^+)^W$. *Then* $\mathrm{cf}(\kappa) > \omega_1$.

These results settle the problem with cofinalities.

Notice that it can be the case that there are two models $W \subseteq V$ which agree on cardinality and such that the least κ which is regular in W and singular in V has uncountable cofinality.

Theorem 15 (Gitik) *There are* $W \subseteq V$ *models of* ZFC *with the same cardinals, the same bounded subsets of* κ *and such that* κ *is regular in* W *and has cofinality* ω_1 *in* V.

Back to the conjecture of Caicedo and Veličković, the best result we can currently prove is the following:

Theorem 16 *Assume $W \subseteq V$ have the same cardinals, V models* MM *and is a set-forcing extension of W. Then $[Ord]^{\leq\omega_1} \subseteq W$.*

Covering properties and the approachability ideal

Covering properties can be used to provide a complete characterization of the approachability ideal $\mathcal{I}[\aleph_{\omega+1}]$ in models of MM.

For example we can prove:

Theorem 17 *Assume:*

- *Martin's maximum holds,*
- $\aleph_\omega$ *is strong limit.*

Then club many points in $\aleph_{\omega+1}$ *of cofinality* $\aleph_n$ *are approachable for all* $n > 1$*.*

This is a partial answer to a problem asked in:

M. Foreman and M. Magidor, A very weak square principle, JSL 1997(1), 175-196.

Is it consistent to have a stationary set of non-approachable points of cofinality $\aleph_2$ in $\aleph_{\omega+1}$?

A result of Magidor covers the case of points of cofinality $\aleph_1$:

Theorem 18 (Magidor) *If Martin's maximum holds, there are stationarily many non-approachable points in $\aleph_{\omega+1}$ of cofinality $\aleph_1$.*

Chang conjectures for singular cardinals

Cummings asked in:

J. Cummings, Collapsing successors of singulars, PAMS, 125(9), 1997, 2703-2709

Is it consistent that $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_2, \aleph_1)$ *for a singular* κ *of countable cofinality?*

Another application of covering properties is the following:

Theorem 19 *Assume* MM. *Then*

$$(\kappa^+, \kappa) \twoheadrightarrow (\aleph_2, \aleph_1)$$

fails for all singular cardinals κ.

The approachability ideal $\mathcal{I}[\lambda]$ has been introduced by Shelah in his analysis of singular cardinals combinatorics.

Three results for $\mathcal{I}[\lambda]$ when λ is the successor of a singular κ:

- There is a stationary set in $\mathcal{I}[\lambda]$. This has been used to prove the existence of scales.

- $\mathcal{I}[\lambda] = P(\lambda)$ unless very large cardinals are behind the scene.

- Preservation of stationarity of S under λ-closed forcing: requires slightly more than $S \in \mathcal{I}[\lambda]$.

An useful characterizaton of the approach-ability ideal

Given

$$d : [\aleph_{\omega+1}]^2 \to \omega$$

- d is normal if
$$D(i,\beta) = \{\alpha < \beta : d(\alpha,\beta) \leq i\}$$
has size less than $\aleph_\omega$ for all i and β,

- d is transitive if whenever $\alpha \in D(i,\beta)$
$$D(i,\alpha) \subseteq D(i,\beta)$$
for all $\alpha \leq \beta$ and i,

- δ of uncountable cofinality λ is d-approachable if there is H unbounded in δ such that $[H]^{<\lambda}$ is covered by the family:
$$\{D(i,\beta) : i < \omega, \beta < \delta\}.$$

The following is a definition of
the ideal $\mathcal{I}[\aleph_{\omega+1}]$ in case that $\aleph_\omega$ is strong limit:

Property 20 *(Shelah) TFAE:*

- $S \in \mathcal{I}[\aleph_{\omega+1}, \aleph_\omega]$,
- *there is a normal and transitive d, and a club C in $\aleph_{\omega+1}$ such that δ is d-approachable for all $\delta \in S \cap C$.*

This characterization of the ideal is not specific for $\aleph_\omega$ and works for all strong limit singular cardinals of any cofinality.

We are led to analyze colorings

$$d : [\aleph_{\omega+1}]^2 \to \omega$$

or equivalently the matrices

$$\mathcal{D}(d) = \{D(i, \beta) : i < \omega, \beta < \aleph_{\omega+1}\}$$

where

$$D(i, \beta) = \{\alpha < \beta : d(\alpha, \beta) \leq i\}$$

We have now a strategy to show that a $\delta < \aleph_{\omega+1}$ of uncountable cofinality $\aleph_n$ is d-approachable:

1. Pick A cofinal subset of δ of minimal order type,

2. take $\mathcal{E}$ to be the transitive collapse of the structure $\{D(i,\alpha) \cap A : i < \omega, \alpha \in A\}$,

3. find G unbounded subset of $\aleph_n$ such that all its initial segments are covered by $\mathcal{E}$,

4. pull back G through the inverse of the transitive collapse of A to an unbounded subset L of A which is covered by
$$\{D(i,\alpha) : i < \omega, \alpha < \delta\},$$
and use the previous property to argue that δ is d-approachable.

the hard work is now only in part 3.

More properties for covering matrices

Back to the previous slide we see that:

$\mathcal{D}(d)$ is an ω-covering matrix on $\aleph_{\omega+1}$ and $\mathcal{E}$ an ω-covering matrix on $\aleph_2$.

- An $\aleph_0$-covering matrix $\mathcal{D}$ on $\aleph_{\omega+1}$ is **transitive** if $\alpha \in D(i,\beta)$ implies $D(i,\alpha) \subseteq D(i,\beta)$.

- $\mathcal{D}$ is **uniform** if for all $\beta < \aleph_{\omega+1}$ there is C club subset of β contained in $D(i,\beta)$ for some $i < \omega$,

- $\beta_{\mathcal{D}} \leq \lambda$ is the least β such that for all i and γ, otp$(D(i,\gamma)) < \beta$,

- $\mathcal{D}$ is normal if $\beta_{\mathcal{D}} < \aleph_{\omega+1}$.

If d is normal and transitive $\mathcal{D}(d)$ is an example of a transitive $\aleph_0$-covering matrix $\mathcal{D}$ on $\aleph_{\omega+1}$ with $\beta_{\mathcal{D}} = \kappa$.

$\mathcal{E}$ is an example of a transitive $\aleph_0$-covering matrix on $\mathrm{cf}(\delta)$ (a priori we can't say much on the value of $\beta_{\mathcal{E}}$).

In this context we are interested in uniform, transitive ω-covering matrices.

Lemma 21 *There is a uniform, normal and transitive ω-covering matrix $\mathcal{D}$ on $\aleph_{\omega+1}$ with $\beta_{\mathcal{D}} = \aleph_\omega$.*

Remark 22 *Consider the previous example of the matrix $\mathcal{E}$ obtained by the transitive collapse of*

$$\{D(i,\alpha) \cap A : i < \omega, \alpha \in A\}$$

where A is a subset of δ of minimal order type.

Provided that A is a club in δ, the matrix $\mathcal{E}$ inherits the property of being a uniform, transitive ω-covering matrix on cf(δ).

What we need to prove is the following:

> *For every unbounded and transitive* $\aleph_0$-*covering matrix* $\mathcal{D}$ *on* $\mathrm{cf}(\delta)$, *there is an unbounded subset of* $\mathrm{cf}(\delta)$ *such that all its initial segments are covered by* $\mathcal{D}$.

$\mathrm{CP}(\mathcal{D})$ is very close to what we are looking for.

It gives us an unbounded subset A of $\mathrm{cf}(\delta)$ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$.

We would like that $[A]^{<\mathrm{cf}(\delta)}$ is covered by $\mathcal{D}$ and not just $[A]^{\aleph_0}$.

The key facts are the following:

Lemma 23 *Assume:*

- *$\mathcal{D}$ is a transitive and uniform $\aleph_0$-covering matrix on $\aleph_n$*

- *Every countable family of stationary subsets of $\aleph_n$ consisting of points of countable cofinality reflects jointly on some $\delta < \aleph_n$.*

Then CP($\mathcal{D}$) *holds i.e. there is A unbounded subset of $\aleph_n$ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$.*

Fact 4 *Assume $\mathcal{D}$ is an $\aleph_0$-covering matrix on $\aleph_n$ and $[A]^{\aleph_0}$ is covered by $\mathcal{D}$. then $[A]^{\aleph_{n-1}}$ is covered by $\mathcal{D}$.*

By the lemma and the fact we get that under MM:

Given a $d : [\aleph_{\omega+1}]^2 \to \aleph_0$ *such that the associated matrix* $D(d)$ *is a normal, transitive and uniform,* $\aleph_0$*-covering matrix on* $\aleph_{\omega+1}$*, every* δ *of cofinality at least* $\aleph_2$ *is* d*-approachable.*

Some open problems (possibly not related to these covering properties....)

Caicedo and Veličković conjecture:

Assume $W \subseteq V$ *have the same cardinals and* V *models* MM. *Do they have the same* ω_1-*sequences of ordinals?*

This conjecture cannot be made false using set-forcing so it should be true!!!

A conjecture in the same spirit is the following

Assume $W \subseteq V$ *are models of* ZFC *with the same cardinals and* V *models* MM. *Then* W *and* V *agrees on cofinality.*

More open problems:

Is it consistent that $S^{\kappa^+}_{\aleph_2}$ *is not in* $\mathcal{I}[\kappa^+]$ *for a* κ *of countable cofinality?*

This is a very large cardinal property and all the known approach to try to achieve this property fail.

Is it consistent with MM *that* $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_2, \aleph_1)$ *for some uncountable* κ*?*

Using classical results on the tree property we can see that $\kappa^{<\kappa} = \kappa$ and PFA imply $(\kappa^+, \kappa) \not\twoheadrightarrow (\aleph_2, \aleph_1)$