Logical Semantics

Logic Colloquium 2008

University of Bern

Monday July 7, 2008 Bern

Plan of the talk

- intersection types
- type theories
- filter models
- Stone duality
- properties of λ-terms

Simple types

$$\sigma ::= \varphi \mid \sigma \to \sigma$$

$$(ax) \quad \Gamma, x : \sigma \vdash x : \sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$(\to I) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

Simple types are preserved by reduction

$$\dfrac{\dfrac{\begin{array}{c}[x:\sigma] \quad \cdots \quad [x:\sigma]\\ \boxed{M^\tau}\\ M:\tau\end{array}}{\lambda x.M:\sigma\to\tau}\,(\to I) \qquad \begin{array}{c}\boxed{N^\sigma}\\ N:\sigma\end{array}}{(\lambda x.M)N:\tau}\,(\to E)
\quad \xRightarrow{\beta\text{-red}} \quad
\begin{array}{c}\begin{array}{ccc}\boxed{N^\sigma} & \vdots & \boxed{N^\sigma}\\ N:\sigma & \cdots & N:\sigma\end{array}\\ \boxed{M[N/x]^\tau}\\ M[N/x]:\tau\end{array}$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$
\begin{array}{c}
\boxed{N^{\sigma}} \\
N : \sigma \\
\boxed{M[N/x]^{\tau}} \\
M[N/x] : \tau
\end{array}
\quad \overset{\beta\text{-exp}}{\Longrightarrow} \quad
\dfrac{\dfrac{\begin{array}{c}[x : \sigma] \\ \boxed{M^{\tau}} \\ M : \tau\end{array}}{\lambda x.M : \sigma \to \tau}\,(\to I) \qquad \begin{array}{c}\boxed{N^{\sigma}} \\ N : \sigma\end{array}}{(\lambda x.M)N : \tau}\,(\to E)
$$

Simple types are NOT preserved by expansion

No occurrences of N

$$\boxed{M^{\tau}}$$

$$\boxed{M[N/x]^{\tau}} \qquad \overset{\beta\text{-exp}}{\Longrightarrow} \qquad \dfrac{M : \tau}{\lambda x.M : \sigma \rightarrow \tau}\ (\rightarrow I) \qquad N : ?$$

$$M[N/x] : \tau \qquad\qquad (\lambda x.M)N$$

Universal type Ω

$$\boxed{M[N/x]^{\tau}} \qquad \stackrel{\beta\text{-exp}}{\Longrightarrow} \qquad \begin{array}{c} [x : \Omega] \\ \boxed{M^{\tau}} \end{array}$$

$$M[N/x] : \tau \qquad\qquad \dfrac{\dfrac{M : \tau}{\lambda x.M : \Omega \to \tau}\,(\to I) \qquad N : \Omega}{(\lambda x.M)N}\,(\to E)$$

Simple types are NOT preserved by expansion

Two or more occurrences of N

$\boxed{N^{\sigma_1}}$ $\boxed{N^{\sigma_2}}$

$N : \sigma_1$ $N : \sigma_2$

$\boxed{M[N/x]^{\tau}}$ $\stackrel{\beta\text{-exp}}{\Longrightarrow}$ $[x : ?]$ $N : ?$

$M[N/x] : \tau$ $(\lambda x.M)N$

Type intersection $\cap$

$$\frac{[x : \sigma_1 \cap \sigma_2]}{x : \sigma_1}\,(\cap E) \qquad \frac{[x : \sigma_1 \cap \sigma_2]}{x : \sigma_2}\,(\cap E)$$

$$\begin{array}{cc} \boxed{N^{\sigma_1}} & \boxed{N^{\sigma_2}} \\ N : \sigma_1 & N : \sigma_2 \end{array}$$

$$\begin{array}{c} \boxed{M[N/x]^{\tau}} \\ M[N/x] : \tau \end{array} \quad \overset{\beta\text{-exp}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\boxed{M^{\tau}} \\ M : \tau\end{array}}{(\lambda x.M) : \sigma_1 \cap \sigma_2 \to \tau}\,(\to I) \qquad \dfrac{\begin{array}{cc}\boxed{N^{\sigma_1}} & \boxed{N^{\sigma_2}} \\ N : \sigma_1 & N : \sigma_2\end{array}}{N : \sigma_1 \cap \sigma_2}\,(\cap I)}{(\lambda x.M)N : \tau}\,(\to E)$$

Intersection types

$$\sigma ::= \varphi \mid \sigma \to \sigma \mid \Omega \mid \sigma \cap \sigma$$

$$(ax) \qquad \Gamma, x:\sigma \vdash x:\sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash M:\sigma \to \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}$$

$$(\to I) \qquad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \to \tau}$$

Intersection types

$$(\Omega) \qquad\qquad \Gamma \vdash M : \Omega$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

$$(\cap I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

$$(\cap I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}$$

$$(\cap E) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}$$

Example

$$\dfrac{\dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma \to \tau}\,(\cap E) \qquad \dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma}\,(\cap E)}{\dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash xx\colon \tau}{\vdash \lambda x.xx\colon (\sigma \to \tau)\cap\sigma \to \tau}\,(\to I)}\,(\to E)$$

Plan of the talk

- intersection types
- type theories
- filter models
- Stone duality
- properties of λ-terms

Type Theories

a type theory ∇ is a set of statements of the form $\sigma \leq_\nabla \tau$ including the following axioms and rules:

$$\sigma \leq \sigma \cap \sigma \qquad \sigma \cap \tau \leq \sigma, \ \sigma \cap \tau \leq \tau$$

$$\sigma \leq \sigma', \ \tau \leq \tau' \Rightarrow \sigma \cap \sigma' \leq \tau \cap \tau' \qquad \sigma \sim \sigma', \ \tau \sim \tau' \Rightarrow \sigma \to \tau \sim \sigma' \to \tau'$$

$$(\sigma \to \tau) \cap (\sigma \to \zeta) \sim \sigma \to \tau \cap \zeta \qquad \sigma \to \Omega \sim \Omega \to \Omega$$

$$\sigma \leq \sigma \qquad \sigma \leq \tau, \ \tau \leq \zeta \ \Rightarrow \sigma \leq \zeta$$

$\sigma \sim \tau$ is short for $\sigma \leq \tau$ and $\tau \leq \sigma$

Subsumption rule

$$\frac{\Gamma \vdash M : \sigma \quad \sigma \leq_\nabla \tau}{\Gamma \vdash M : \tau} \, (\leq_\nabla)$$

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$$\frac{\Gamma \vdash M : \sigma \quad \sigma \leq_\nabla \tau}{\Gamma \vdash M : \tau} \; (\leq_\nabla)$$

$$\dfrac{\dfrac{x : \sigma \to \tau \cap \zeta \vdash x : \sigma \to \tau \cap \zeta}{x : \sigma \to \tau \cap \zeta \vdash x : (\sigma \to \tau) \cap (\sigma \to \zeta)} \; (\leq_\nabla)}{\vdash \lambda x.x : (\sigma \to \tau \cap \zeta) \to (\sigma \to \tau) \cap (\sigma \to \zeta)} \; (\to I)$$

Subsumption rule

$$\frac{\Gamma \vdash M : \sigma \quad \sigma \leq_\nabla \tau}{\Gamma \vdash M : \tau}\ (\leq_\nabla)$$

$$\cfrac{\cfrac{x : \sigma \rightarrow \tau \cap \zeta \vdash x : \sigma \rightarrow \tau \cap \zeta}{x : \sigma \rightarrow \tau \cap \zeta \vdash x : (\sigma \rightarrow \tau) \cap (\sigma \rightarrow \zeta)}\ (\leq_\nabla)}{\vdash \lambda x.x : (\sigma \rightarrow \tau \cap \zeta) \rightarrow (\sigma \rightarrow \tau) \cap (\sigma \rightarrow \zeta)}\ (\rightarrow I)$$

$$\Gamma \vdash^\nabla M : \tau$$

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The set $\mathcal{F}^{\nabla}$ of filters

A ∇-filter is a set X of intersection types such that:

- $\Omega \in X$
- if $\sigma \leq_{\nabla} \tau$ and $\sigma \in X$, then $\tau \in X$
- if $\sigma, \tau \in X$, then $\sigma \cap \tau \in X$

$\mathcal{F}_{\nabla}$ is the set of filters

$\uparrow^{\nabla} X$ is the filter generated by X

$\uparrow^{\nabla} \sigma$ is $\uparrow^{\nabla} \{\sigma\}$

$\langle \mathcal{F}_{\nabla}, \subseteq \rangle$ is an ω-algebraic complete lattice

$$
\begin{tikzcd}
 & \mathcal{T} \arrow[d, dotted] & \\
 & \uparrow^{\nabla}(X \cup Y) \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & X \cap Y \arrow[d, dotted] & \\
 & \uparrow^{\nabla}\Omega &
\end{tikzcd}
$$

Interpretation of λ-terms

For any λ-term M and environment

$$\rho : \mathtt{var} \to \mathcal{F}_\nabla$$

$$[\![M]\!]_\rho^{\mathcal{F}_\nabla} = \{\tau \in \mathcal{T} \mid \exists \Gamma \models \rho.\ \Gamma \vdash M : \tau\}$$

where $\Gamma \models \rho$ iff $(x : \sigma) \in \Gamma$ implies $\sigma \in \rho(x)$.

Interpretation of λ-terms

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Is $\langle \mathcal{F}_\nabla, \subseteq \rangle$ a λ-model (filter model)?

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where $\Gamma \models \rho$ iff $(x : \sigma) \in \Gamma$ implies $\sigma \in \rho(x)$.

Is $\langle \mathcal{F}_{\nabla}, \subseteq \rangle$ a λ-model (filter model)?

iff $\Gamma \vdash^{\nabla} \lambda x.M : \sigma \rightarrow \tau$ implies

$$\Gamma, x : \sigma \vdash^{\nabla} M : \tau$$

Filter Models

With suitable type theories we can obtain filter models isomorphic to

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Filter Models

With suitable type theories we can obtain filter models isomorphic to

- Scott inverse limit models;
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- Abramsky-Ong model;
- Girard qualitative model;
-
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With suitable type theories we can obtain filter models isomorphic to

- Scott inverse limit models;
- Scott P_ω model;
- Plotkin-Engeler models;
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- Girard quantitative model;
- $\ldots$

Filter Models

With suitable type theories we can obtain filter models isomorphic to

- Scott inverse limit models;
- Scott P_ω model;
- Plotkin-Engeler models;
- Abramsky-Ong model;
- Girard qualitative model;
- Girard quantitative model;
- ...

Filter models versus inverse limit models

We can describe an inverse limit model $\mathcal{D}_\infty$ by taking:

- the types freely generated by closing (a set of atomic types corresponding to) the elements of $\mathcal{D}_0$ under the function type constructor $\rightarrow$ and the intersection type constructor $\cap$;

model

Filter models versus inverse limit models

We can describe an inverse limit model $\mathcal{D}_\infty$ by taking:

- the types freely generated by closing (a set of atomic types corresponding to) the elements of $\mathcal{D}_0$ under the function type constructor $\rightarrow$ and the intersection type constructor $\cap$;
- the preorder between types induced by reversing the order in $\mathcal{D}_0$ and by encoding the initial projection, according to the correspondence:

function type constructor	$\mapsto$	step function
intersection type constructor	$\mapsto$	join

model

Example

$$\begin{tikzcd} \hat{\mathsf{n}} \arrow[d] \\ \mathsf{n} \arrow[d] \\ \bot \end{tikzcd}$$

$$\mathcal{D}_0$$

$$\hat{\nu} \leq \nu \leq \Omega$$

Example

$$\begin{array}{c} \hat{\mathsf{n}} \\ \downarrow \\ \mathsf{n} \\ \downarrow \\ \bot \end{array} \qquad \begin{array}{c} \bot \Rightarrow \hat{\mathsf{n}} \\ \vdots \\ \mathsf{n} \Rightarrow \hat{\mathsf{n}} \\ \vdots \\ \hat{\mathsf{n}} \Rightarrow \mathsf{n} \\ \vdots \\ \bot \Rightarrow \bot \end{array}$$

$$\mathcal{D}_0 \qquad [\mathcal{D}_0 \longrightarrow \mathcal{D}_0]$$

$$\hat{\nu} \leq \nu \leq \Omega$$

Example

$$\begin{array}{c} \hat{\mathsf{n}} \\ \downarrow \\ \mathsf{n} \\ \downarrow \\ \perp \end{array} \qquad\qquad \begin{array}{c} \perp \Rightarrow \hat{\mathsf{n}} \\ \vdots \\ \mathsf{n} \Rightarrow \hat{\mathsf{n}} \\ \vdots \\ \hat{\mathsf{n}} \Rightarrow \mathsf{n} \\ \vdots \\ \perp \Rightarrow \perp \end{array}$$

$\mathcal{D}_0$

$[\mathcal{D}_0 \rightarrow \mathcal{D}_0]$

$\hat{\nu} \leq \nu \leq \Omega$

$\mathsf{i}_0(\hat{\mathsf{n}}) = \mathsf{n} \Rightarrow \hat{\mathsf{n}}$ $\quad$ $\mathsf{i}_0(\mathsf{n}) = \hat{\mathsf{n}} \Rightarrow \mathsf{n}$ $\quad$ $\mathsf{i}_0(\perp) = \perp \Rightarrow \perp$

$\hat{\nu} \sim \nu \rightarrow \hat{\nu}$ $\quad$ $\nu \sim \hat{\nu} \rightarrow \nu$ $\quad$ $\Omega \sim \Omega \rightarrow \Omega$

properties

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Stone dualities

we started from types and arrived to models:
what is the framework?

Stone dualities

we started from types and arrived to models:
what is the framework?

Stone dualities

Stone dualities

we started from types and arrived to models:
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Stone dualities

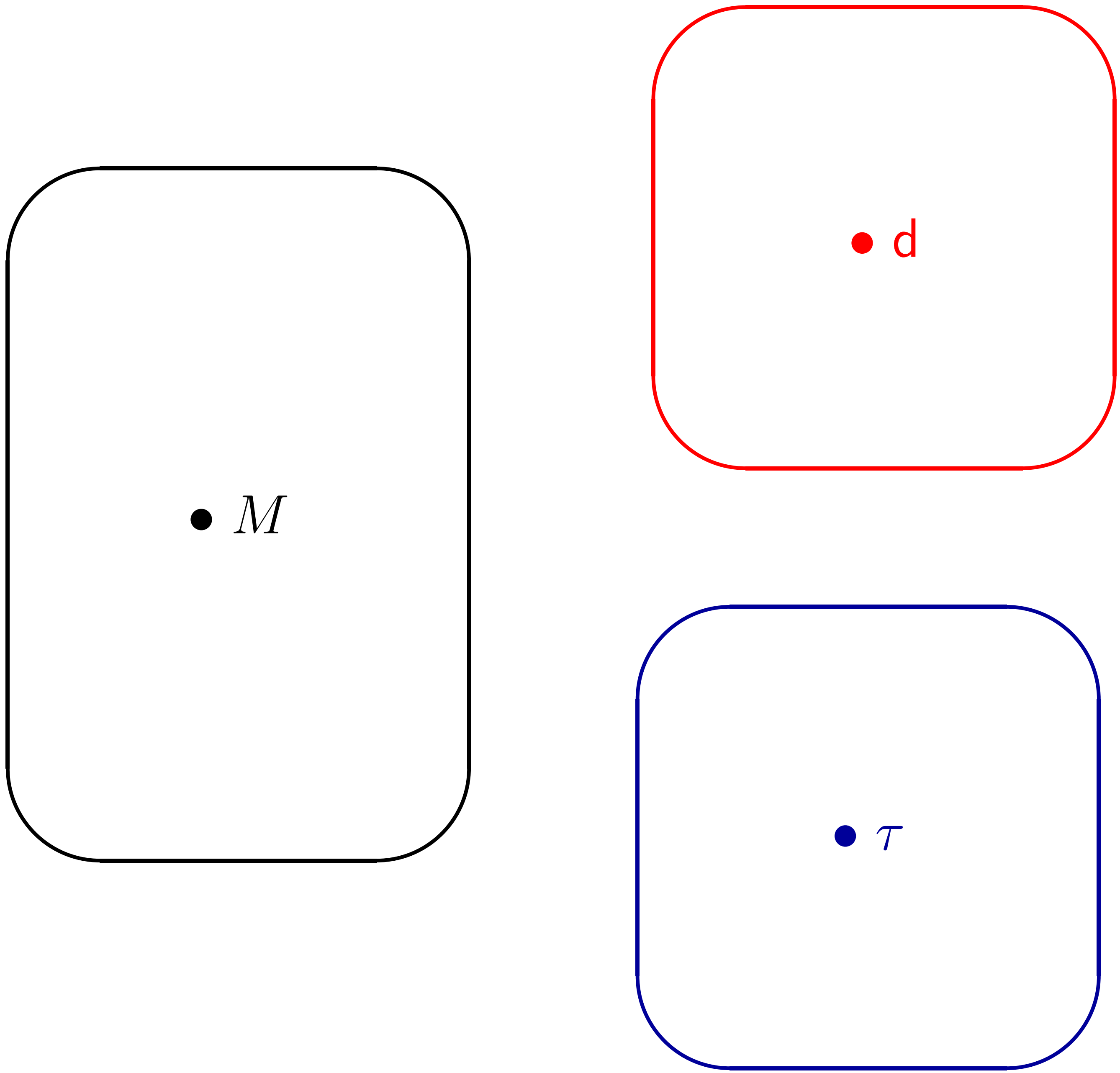
topological spaces as partial orders

Stone spaces as Boolean algebras (Stone, 36)

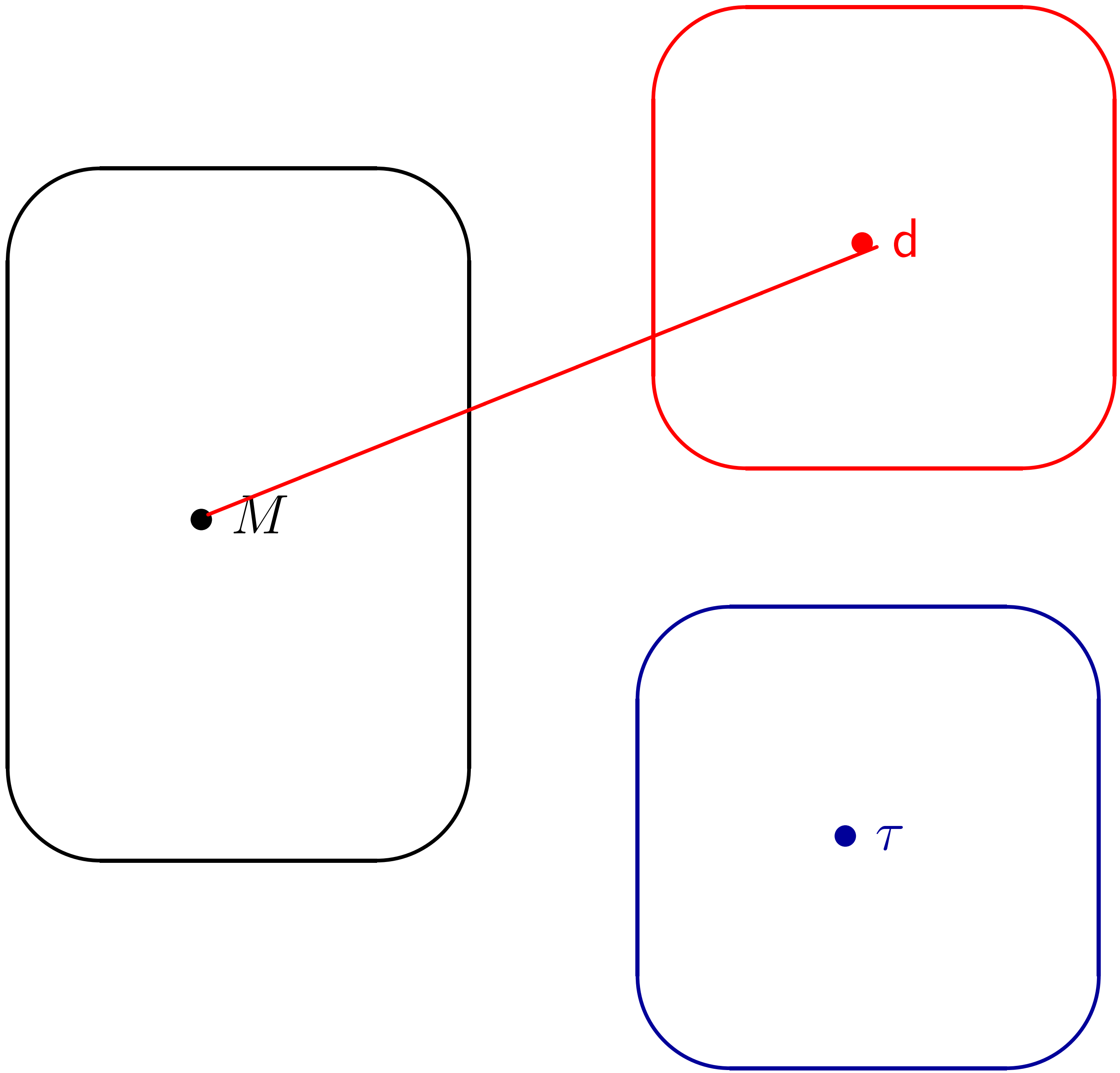
Scott domains as information systems (Scott, 82)

ω-algebraic complete lattices as intersection type theories
(Coppo et al., 84)

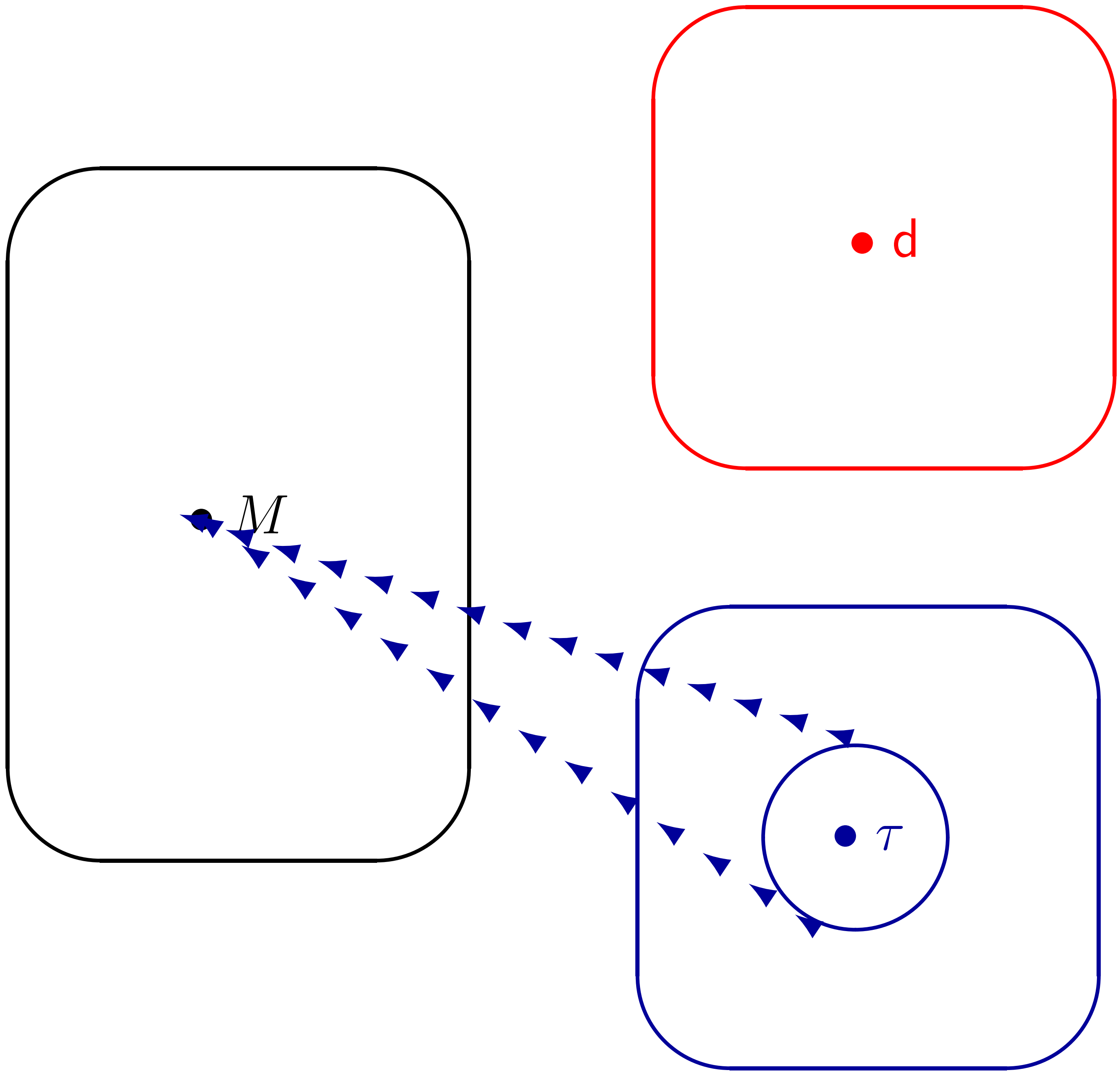
SFP domains as pre-locales (Abramsky, 91)


d
M
τ

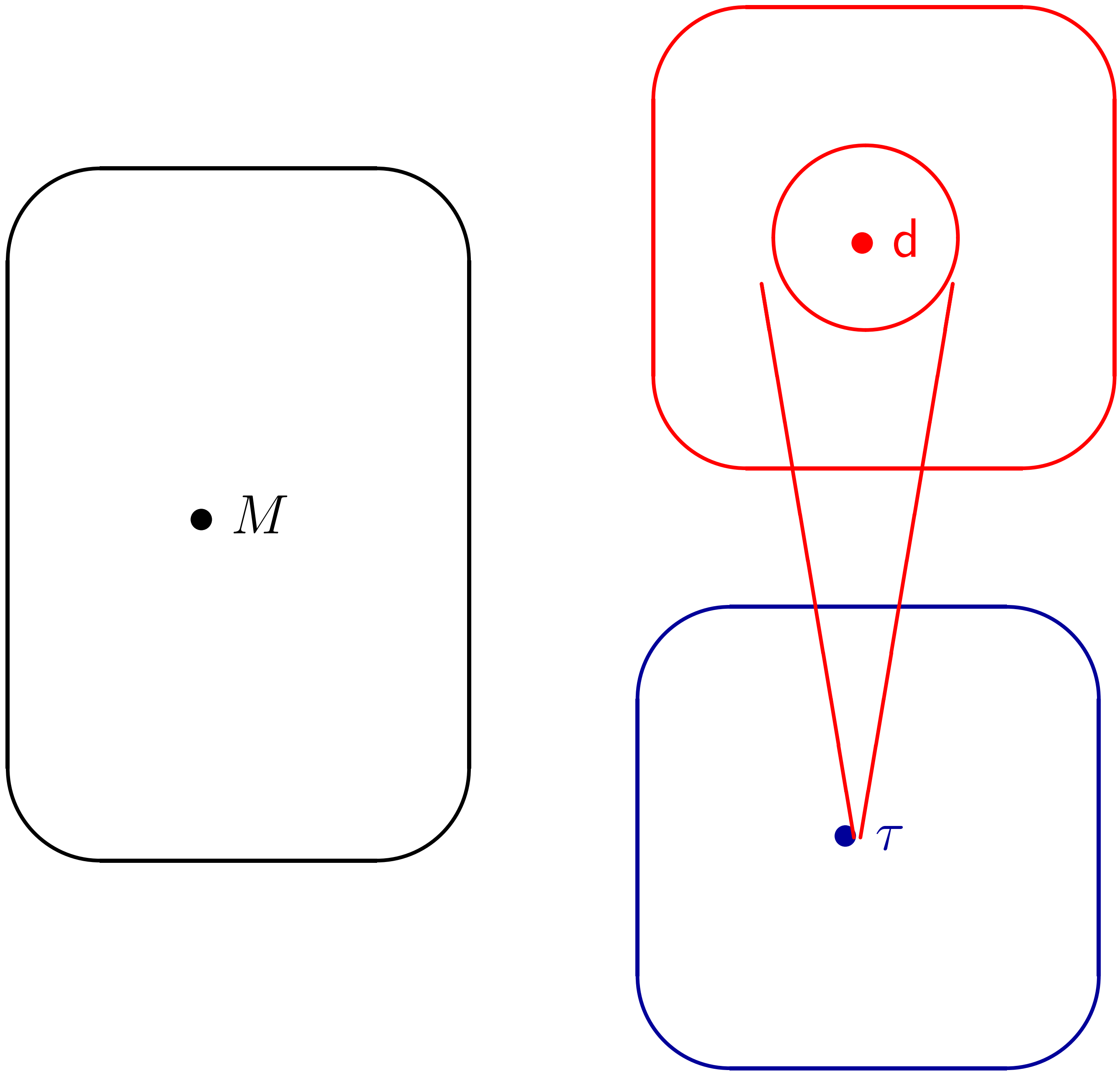
semantics of terms



type assignment



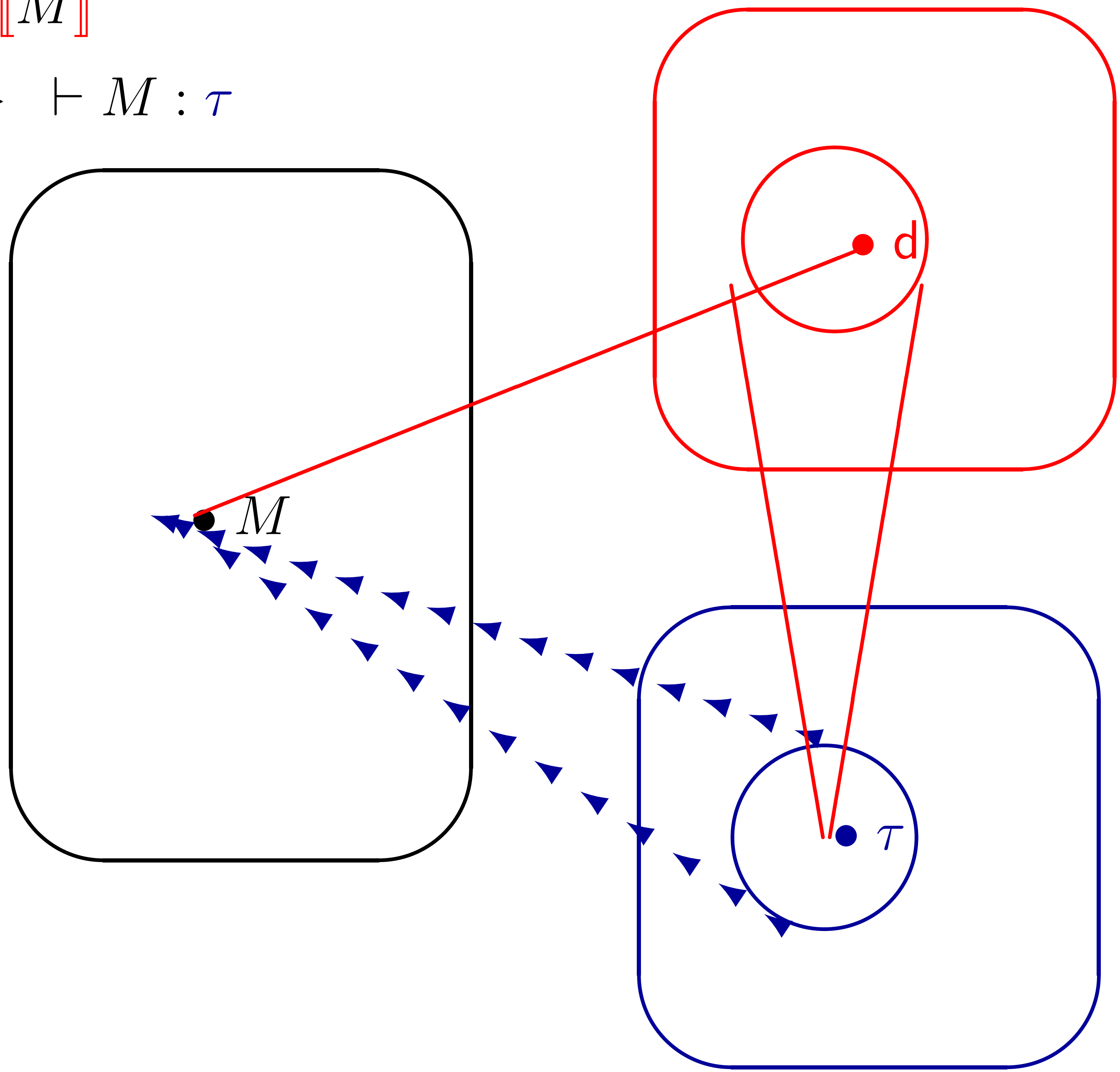
semantics of types



Stone duality

$\mathsf{d} = [\![M]\!]$

$\mathsf{d} \in [\![\tau]\!] \;\Leftrightarrow\; \vdash M : \tau$



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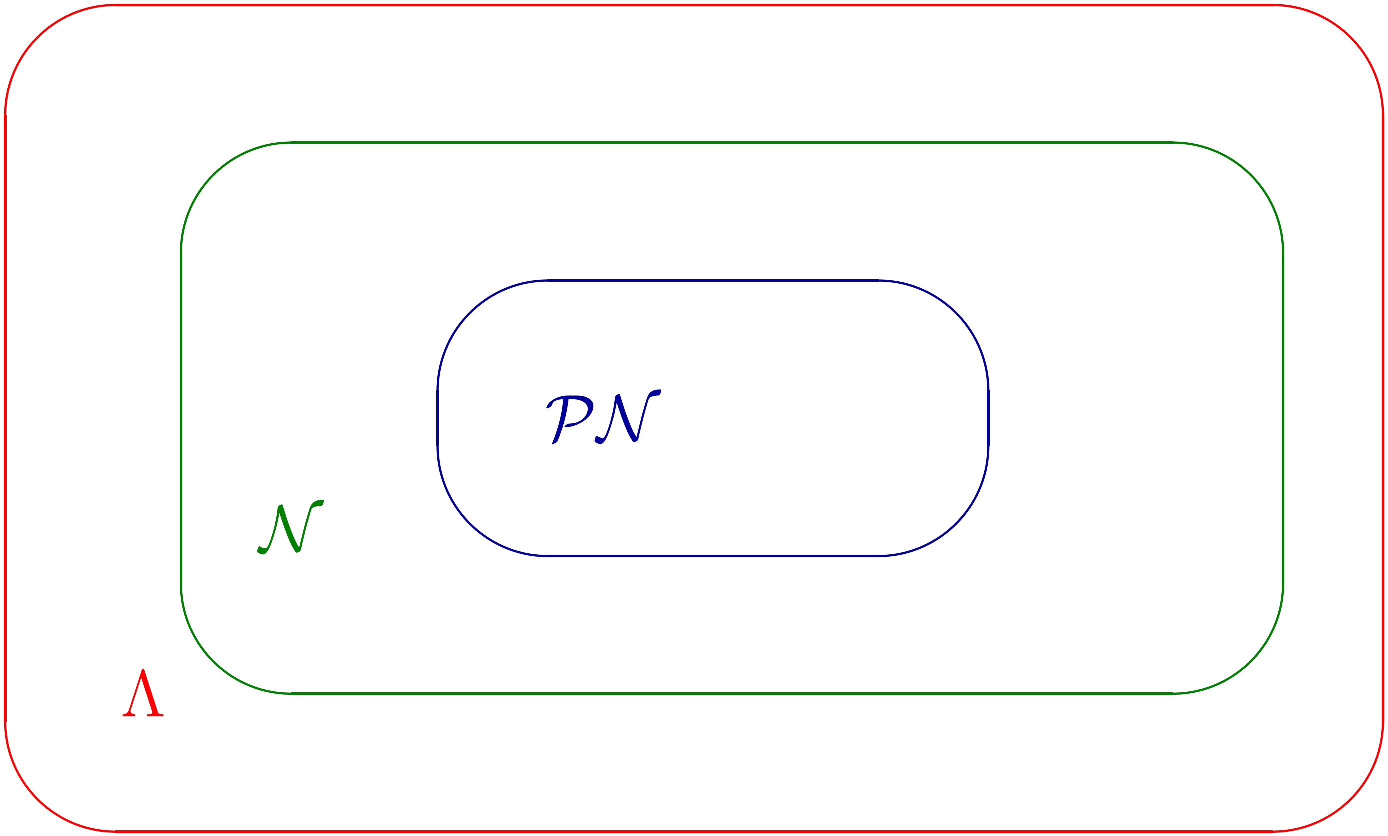
A filter model characterizing normalising terms

$M \in \mathcal{N}$ iff $M \twoheadrightarrow_\beta$ a normal form

$M \in \mathcal{PN}$ iff $\forall \vec{N} \in \mathcal{N}\ M\vec{N} \in \mathcal{N}$



$\hat{\nu} \leq \nu \leq \Omega$

A filter model characterizing normalising terms

$$M \in \mathcal{PN} \text{ implies } \forall N \in \mathcal{N}\ MN \in \mathcal{PN}$$

$$M : \hat{\nu} \text{ implies } \forall N : \nu\ MN : \hat{\nu}$$

$$\hat{\nu} \sim \nu \to \hat{\nu}$$

A filter model characterizing normalising terms

$$M \in \mathcal{PN} \text{ implies } \forall N \in \mathcal{N} \; MN \in \mathcal{PN}$$

$$M : \hat{\nu} \text{ implies } \forall N : \nu \; MN : \hat{\nu}$$

$$\hat{\nu} \sim \nu \to \hat{\nu}$$

$$M \in \mathcal{N} \text{ implies } \forall N \in \mathcal{PN} \; MN \in \mathcal{N}$$

$$M : \nu \text{ implies } \forall N : \hat{\nu} \; MN : \nu$$

$$\nu \sim \hat{\nu} \to \nu$$

A filter model characterizing normalising terms

$$M \in \mathcal{PN} \text{ implies } \forall N \in \mathcal{N}\ MN \in \mathcal{PN}$$

$$M : \hat{\nu} \text{ implies } \forall N : \nu\ MN : \hat{\nu}$$

$$\hat{\nu} \sim \nu {\rightarrow} \hat{\nu}$$

$$M \in \mathcal{N} \text{ implies } \forall N \in \mathcal{PN}\ MN \in \mathcal{N}$$

$$M : \nu \text{ implies } \forall N : \hat{\nu}\ MN : \nu$$

$$\nu \sim \hat{\nu} {\rightarrow} \nu$$

$$M \in \Lambda \text{ implies } \forall N \in \Lambda\ MN \in \Lambda$$

$$M : \Omega \text{ implies } \forall N : \Omega\ MN : \Omega$$

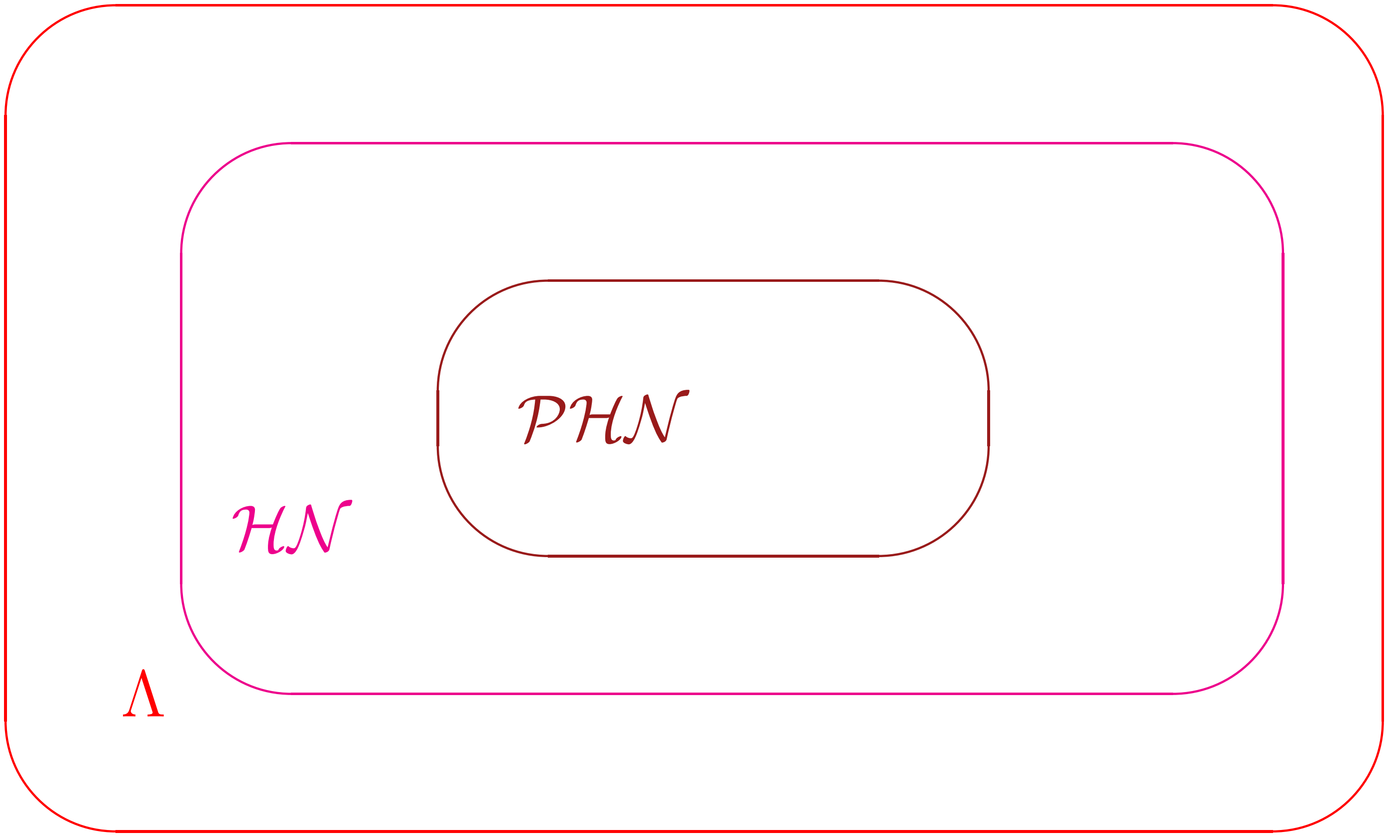
$$\Omega \sim \Omega {\rightarrow} \Omega$$

isomorphism

A filter model characterizing head normalising terms

$$M \in \mathcal{HN} \text{ iff } M \twoheadrightarrow_\beta \lambda \vec{x}.y\vec{N}$$

$$M \in \mathcal{PHN} \text{ iff } \forall \vec{N} \in \Lambda \; M\vec{N} \in \mathcal{HN}$$



$$\hat{\mu} \leq \mu \leq \Omega$$

A filter model characterizing head normalising terms

$$M \in \mathcal{PHN} \text{ implies } \forall N \in \Lambda\ MN \in \mathcal{PHN}$$

$$M : \hat{\mu} \text{ implies } \forall N : \Omega\ MN : \hat{\mu}$$

$$\hat{\mu} \sim \Omega \to \hat{\mu}$$

A filter model characterizing head normalising terms

$$M \in \mathcal{PHN} \text{ implies } \forall N \in \Lambda \; MN \in \mathcal{PHN}$$

$$M : \hat{\mu} \text{ implies } \forall N : \Omega \; MN : \hat{\mu}$$

$$\hat{\mu} \sim \Omega \to \hat{\mu}$$

$$M \in \mathcal{HN} \text{ implies } \forall N \in \mathcal{PHN} \; MN \in \mathcal{HN}$$

$$M : \mu \text{ implies } \forall N : \hat{\mu} \; MN : \mu$$

$$\mu \sim \hat{\mu} \to \mu$$

A filter model characterizing head normalising terms

$$M \in \mathcal{PHN} \text{ implies } \forall N \in \Lambda \; MN \in \mathcal{PHN}$$

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$$M : \mu \text{ implies } \forall N : \hat{\mu} \; MN : \mu$$

$$\mu \sim \hat{\mu} \rightarrow \mu$$

$$M \in \Lambda \text{ implies } \forall N \in \Lambda \; MN \in \Lambda$$

$$M : \Omega \text{ implies } \forall N : \Omega \; MN : \Omega$$

$$\Lambda \sim \Lambda \rightarrow \Lambda$$

Intersection types characterize also

closable terms

Intersection types characterize also

closable terms

terms of the $\mathbf{I}$-calculus

Intersection types characterize also

closable terms

terms of the **I**-calculus

weak head normalising terms

Intersection types characterize also

closable terms

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. . .

Intersection types characterize also

closable terms

terms of the $\mathbf{I}$-calculus

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. . .

A filter model characterising...

$$
\begin{tikzcd}
 & \Lambda & & \\
 & \mathcal{WN} \arrow[u] & & \\
\mathcal{PWN} \arrow[ur] & & \mathcal{HN} \arrow[ul] & \\
 & \mathcal{PHN} \arrow[ul] \arrow[ur] & & \mathcal{N} \arrow[ul] \\
 & & \mathcal{PN} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Inverse limit model

$$
\begin{tikzcd}[column sep=small]
 & \hat{n} \arrow[dl] \arrow[dr] & \\
\hat{h} \arrow[dr] & & n \arrow[dl] \\
 & h \arrow[d] & \\
 & \bot &
\end{tikzcd}
$$

$$\mathcal{D}_0$$

$$
\begin{tikzcd}[column sep=tiny]
 & \bot \Rightarrow \hat{n} \arrow[d, dotted] & \\
 & (\bot \Rightarrow \hat{h}) \sqcup (n \Rightarrow \hat{n}) \arrow[dl, dotted] \arrow[dr, dotted] & \\
\bot \Rightarrow \hat{h} \arrow[dr, dotted] & & (\hat{h} \Rightarrow h) \sqcup (\hat{n} \Rightarrow n) \arrow[dl, dotted] \\
 & \hat{h} \Rightarrow h \arrow[d, dotted] & \\
 & \bot \Rightarrow \bot \arrow[d] & \\
 & \bot &
\end{tikzcd}
$$

$$[\mathcal{D}_0 \to \mathcal{D}_0]_\bot$$

$$i_0(\hat{n}) = (\bot \Rightarrow \hat{h}) \sqcup (n \Rightarrow \hat{n}) \qquad i_0(n) = (\hat{h} \Rightarrow h) \sqcup (\hat{n} \Rightarrow n)$$

$$i_0(\hat{h}) = \bot \Rightarrow \hat{h} \qquad i_0(h) = \hat{h} \Rightarrow h \qquad i_0(\bot) = \bot$$

Types and preorder

$$
\begin{tikzcd}
 & \hat{n} \arrow[dl] \arrow[dr] & \\
\hat{h} \arrow[dr] & & n \arrow[dl] \\
 & h \arrow[d] & \\
 & \perp &
\end{tikzcd}
$$

correspondence

Types and preorder

$$
\begin{tikzcd}
 & \hat{\mathsf{n}} \arrow[dl] \arrow[dr] & \\
\hat{\mathsf{h}} \arrow[dr] & & \mathsf{n} \arrow[dl] \\
 & \mathsf{h} \arrow[d] & \\
 & \bot &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \hat{\nu} & \\
\hat{\mu} \arrow[ur] & & \nu \arrow[ul] \\
 & \mu \arrow[ul] \arrow[ur] & \\
 & \Omega \arrow[u] &
\end{tikzcd}
$$

correspondence

Types and preorder

$$
\begin{tikzcd}
 & \hat{\mathsf{n}} \arrow[dl] \arrow[dr] & \\
\hat{\mathsf{h}} \arrow[dr] & & \mathsf{n} \arrow[dl] \\
 & \mathsf{h} \arrow[d] & \\
 & \bot &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \hat{\nu} & \\
\hat{\mu} \arrow[ur] & & \nu \arrow[ul] \\
 & \mu \arrow[ul] \arrow[ur] & \\
 & \Omega \arrow[u] &
\end{tikzcd}
$$

$$\mathsf{i}_0(\hat{\mathsf{n}}) = (\bot \Rightarrow \hat{\mathsf{h}}) \sqcup (\mathsf{n} \Rightarrow \hat{\mathsf{n}}) \qquad \mathsf{i}_0(\mathsf{n}) = (\hat{\mathsf{h}} \Rightarrow \mathsf{h}) \sqcup (\hat{\mathsf{n}} \Rightarrow \mathsf{n})$$

$$\mathsf{i}_0(\hat{\mathsf{h}}) = \bot \Rightarrow \hat{\mathsf{h}} \qquad \mathsf{i}_0(\mathsf{h}) = \hat{\mathsf{h}} \Rightarrow \mathsf{h}$$

correspondence

Types and preorder

$$
\begin{tikzcd}
 & \hat{\mathsf{n}} \arrow[dl] \arrow[dr] & \\
\hat{\mathsf{h}} \arrow[dr] & & \mathsf{n} \arrow[dl] \\
 & \mathsf{h} \arrow[d] & \\
 & \bot &
\end{tikzcd}
\qquad\qquad
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 & \hat{\nu} & \\
\hat{\mu} \arrow[ur] & & \nu \arrow[ul] \\
 & \mu \arrow[ul] \arrow[ur] & \\
 & \Omega \arrow[u] &
\end{tikzcd}
$$

$$\mathsf{i}_0(\hat{\mathsf{n}}) = (\bot \Rightarrow \hat{\mathsf{h}}) \sqcup (\mathsf{n} \Rightarrow \hat{\mathsf{n}}) \qquad \mathsf{i}_0(\mathsf{n}) = (\hat{\mathsf{h}} \Rightarrow \mathsf{h}) \sqcup (\hat{\mathsf{n}} \Rightarrow \mathsf{n})$$

$$\mathsf{i}_0(\hat{\mathsf{h}}) = \bot \Rightarrow \hat{\mathsf{h}} \qquad \mathsf{i}_0(\mathsf{h}) = \hat{\mathsf{h}} \Rightarrow \mathsf{h}$$

$$\hat{\nu} \sim (\Omega \to \hat{\mu}) \cap (\nu \to \hat{\nu}) \qquad \nu \sim (\hat{\mu} \to \mu) \cap (\hat{\nu} \to \nu)$$

$$\hat{\mu} \sim \Omega \to \hat{\mu} \qquad \mu \sim \hat{\mu} \to \mu$$

correspondence

Main Theorem

$$M \in \mathcal{PN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \hat{\nu}$$

$$M \in \mathcal{N} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \nu$$

$$M \in \mathcal{PHN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \hat{\mu}$$

$$M \in \mathcal{HN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \mu$$

$$M \in \mathcal{PWN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \Omega^n \to \Omega \text{ for all } n \in \mathbb{N}$$

$$M \in \mathcal{WN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \Omega \to \Omega$$

$\Gamma_{\hat{\nu}} = \{x : \hat{\nu} \mid \forall x \in \texttt{var}\}$

Open problems

- can we characterize in some significant way the class of evaluation properties which we can characterize using filter models?
- is there a method for going from a logical specification of a property to the appropriate filter model?
- which properties cannot be characterised in the same filter model?
- which properties must be characterised in the same filter model?

questions

questions

thank you for your attention

Klop's extended λ-calculus $\Lambda^\star$

$\lambda^\star$-terms

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$$S ::= x \mid \lambda x.S \mid SS \mid [S, S]$$

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$\overrightarrow{\kappa}$

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$\xrightarrow[\kappa]{}$

$$[\lambda x.S, U_1, \ldots, U_n]T \;\xrightarrow[\kappa]{}\; [S[x := T], U_1, \ldots, U_n]$$
if $x \in \mathsf{FV}(S)$

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$\underset{\kappa}{\longrightarrow}$

$$[\lambda x.S, U_1, \ldots, U_n]T \underset{\kappa}{\longrightarrow} [S[x := T], U_1, \ldots, U_n]$$

if $x \in$ FV(S)

$$[\lambda x.S, U_1, \ldots, U_n]T \underset{\kappa}{\longrightarrow} [S, U_1, \ldots, U_n, T]$$

if $x \notin$ FV(S)

An inverse limit model for $\mathcal{SN}^{\star}$ and $\mathcal{PSN}^{\star}$

strict function $f(\perp) = \perp$

An inverse limit model for $\mathcal{SN}^\star$ and $\mathcal{PSN}^\star$

strict function $f(\perp) = \perp$

$$\begin{array}{c} \hat{\mathsf{s}} \\ \downarrow \\ \mathsf{s} \\ \downarrow \\ \perp \end{array}$$

$\mathcal{D}_0$

An inverse limit model for $\mathcal{SN}^\star$ and $\mathcal{PSN}^\star$

strict function $f(\bot) = \bot$

$$\begin{tikzcd} \hat{\mathsf{s}} \arrow[d] \\ \mathsf{s} \arrow[d] \\ \bot \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathsf{s} \Rightarrow \hat{\mathsf{s}} \arrow[d, dotted] \\ \hat{\mathsf{s}} \Rightarrow \mathsf{s} \arrow[d, dotted] \\ \bot \Rightarrow \bot \end{tikzcd}$$

$\mathcal{D}_0$ $\qquad\qquad$ $[\mathcal{D}_0 \longrightarrow_\bot \mathcal{D}_0]$

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strict function $f(\perp) = \perp$

$$
\begin{tikzcd}
\hat{\mathsf{s}} \arrow[d] \\
\mathsf{s} \arrow[d] \\
\perp
\end{tikzcd}
\qquad\qquad\qquad
\begin{tikzcd}
\mathsf{s}\Rightarrow\hat{\mathsf{s}} \arrow[d, dotted] \\
\hat{\mathsf{s}}\Rightarrow\mathsf{s} \arrow[d, dotted] \\
\perp\Rightarrow\perp
\end{tikzcd}
$$

$\mathcal{D}_0$ $\qquad\qquad$ $[\mathcal{D}_0 \to_\perp \mathcal{D}_0]$

$S \in \mathcal{PSN}^\star, T \in \mathcal{SN}^\star$ implies $ST \in \mathcal{PSN}^\star$

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\mathsf{s} \arrow[d] & \hat{\mathsf{s}}\Rightarrow\mathsf{s} \arrow[d, dotted] \\
\bot & \bot\Rightarrow\bot
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$$
\begin{tikzcd}
\hat{\mathsf{s}} \arrow[d] & & & \mathsf{s} \Rightarrow \hat{\mathsf{s}} \arrow[d, dotted] \\
\mathsf{s} \arrow[d] \arrow[rrr] & & & \hat{\mathsf{s}} \Rightarrow \mathsf{s} \arrow[d, dotted] \\
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$$\begin{array}{c} \hat{s} \\ \downarrow \\ s \\ \downarrow \\ \bot \end{array} \qquad\qquad\qquad \begin{array}{c} s \Rightarrow \hat{s} \\ \vdots \\ \downarrow \\ \hat{s} \Rightarrow s \\ \vdots \\ \downarrow \\ \bot \Rightarrow \bot \end{array}$$

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$S \in \Lambda^\star, T \in \Lambda^\star$ implies $ST \in \Lambda^\star$ $\qquad$ $\mathsf{i}_0^{\mathcal{S}}(\perp) = \perp \Rightarrow \perp$

Simple easy terms

A filter scheme $\mathcal{S}$ is a mapping: $(\nabla, \zeta) \to \nabla'$

A lambda term E is simple easy if there exists a filter scheme $\mathcal{S}_E$ such that

$$[\![E]\!]^{\nabla'} = \uparrow^{\nabla'} \zeta \sqcup [\![E]\!]^{\nabla}$$

where

- $\nabla' = \mathcal{S}_E(\nabla, \zeta)$
- ∇' induces a filter model whenever ∇ induces a filter model

Some simple easy terms

$$\mathbf{W}_* \equiv \lambda x.xx \qquad \mathbf{W_2} \equiv \mathbf{W}_*\mathbf{W}_* \qquad \mathbf{I} \equiv \lambda x.x$$

$$\begin{array}{lrcl}
\mathbf{W_2} & \mathcal{S}_{\mathbf{W_2}}(\nabla,\zeta) & = & \nabla \cup \{\varphi \sim \varphi{\to}\zeta\} \\
\mathbf{W_2W_2} & \mathcal{S}_{\mathbf{W_2W_2}}(\nabla,\zeta) & = & \nabla \cup \{\psi \sim (\psi{\to}\omega) \cap (\psi{\to}\omega{\to}\zeta), \\
 & & & \omega \sim \phi{\to}\omega, \phi \sim \omega{\to}\omega\} \\
\mathbf{W_2I} & \mathcal{S}_{\mathbf{W_2I}}(\nabla,\zeta) & = & \nabla \cup \{\alpha \sim (\gamma{\to}\alpha) \cap (\beta{\to}\delta{\to}\zeta), \\
 & & & \beta \sim \gamma{\to}\alpha, \gamma \sim \gamma{\to}\beta, \\
 & & & \delta \sim \eta{\to}\eta, \eta \sim \delta{\to}\eta, \\
 & & & \alpha \leq \beta \leq \gamma\}
\end{array}$$

Key property of simple easy terms

If E is a simple easy term then there exists a filter model $\mathcal{F}_\nabla$ in which $[\![E]\!]^\nabla = \uparrow^\nabla \{\sigma \in \mathcal{T} \mid \mathrm{P}(\sigma)\}$ where P is a "continuous" predicate on types.

Semantic proofs of easiness

a closed term E is easy if, for any other closed term M, the theory $\lambda\beta + \{M = E\}$ is consistent

all simple easy terms are easy

Consistency of λ-theories

The λ-theory $\mathcal{J}$ is axiomatized by

$$\mathbf{W_2}xx = x; \quad \mathbf{W_2}xy = \mathbf{W_2}yx$$
$$\mathbf{W_2}x(\mathbf{W_2}yz) = \mathbf{W_2}(\mathbf{W_2}xy)z$$

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idea: interpret $\mathbf{W_2}$ as the join operator on filters

the join operator on filters is a filter generated by a continuous predicate

$$U = \uparrow^{\nabla} \{\sigma \to \tau \to \sigma \cap \tau\}$$
$$\mathsf{P}(\Sigma^{\nabla}, \rho) \quad \Leftrightarrow \rho \equiv \sigma \to \tau \to \sigma \cap \tau$$

Minimal fixed point operator

For all simple easy terms E there are filter models $\mathcal{F}^{\nabla}$ such that $[\![E]\!]^{\nabla}$ represents the minimal fixed point operator fix

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Warning: there is no fixed point combinator $\tilde{\mathrm{Y}}$ such that $[\![\tilde{\mathrm{Y}}]\!]^{\nabla}$ represents fix in *each* filter model $\mathcal{F}^{\nabla}$

Counter-example: Park model where the interpretation of each closed term (included $\tilde{\mathrm{Y}}\mathrm{I}$) is greater than $\perp$

Filter models for recursive terms

$\Lambda\mu$-terms:

$$M ::= x \mid MM \mid \lambda x.M \mid \mu x.M$$

reduction rule:

$$\mu x.M \to M[\mu x.M/x]$$

Filter models for recursive terms

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reduction rule:

$$\mu x.M \to M[\mu x.M/x]$$

For all simple easy terms E there are filter models $\mathcal{F}^{\nabla}$ such that

$$[\![E(\lambda x.M)]\!]^{\nabla} = [\![\mu x.M]\!]^{\nabla}$$

for all $M \in \Lambda\mu$

Open problems

- does easiness imply simple easiness?
- which λ-theories can be proved consistent using the present approach?
- which operators can be equated to simple easy terms?