Three Infinitesimalist Theories of Continua

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Paul du Bois-Reymond (1870-1882), Giuseppe Veronese (1891), and Charles Sanders Peirce (1898-1900) all proposed non-standard theories of continua that make use of infinitesimals. Du Bois-Reymond's and Veronese's theories were given precision by Felix Hausdorff (1909) and Tullio Levi-Civita (1898), respectively, and in [Ehrlich forthcoming] we provided a formal replacement for the purported continuum of Peirce.

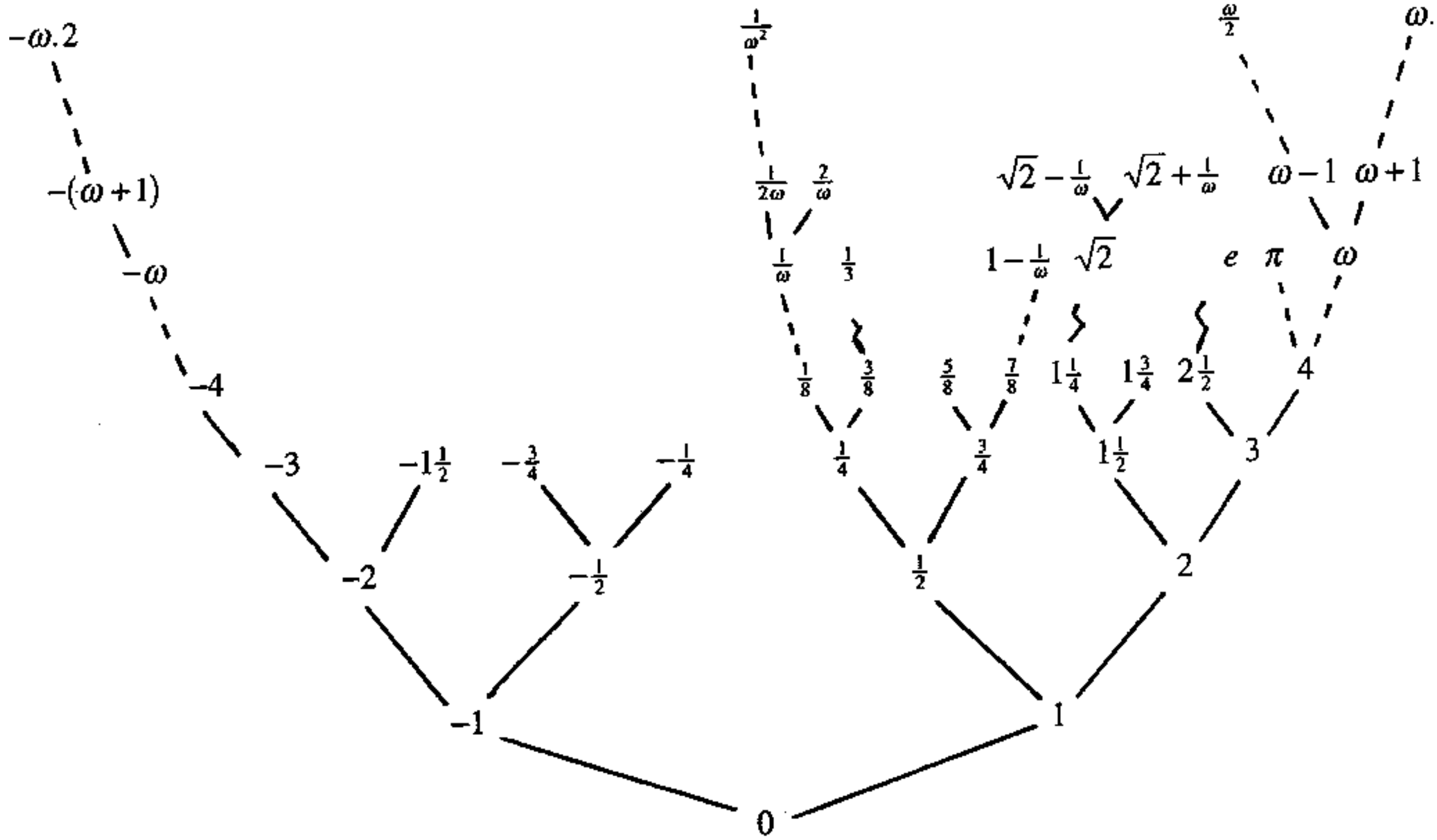
In a number of papers we have suggested that whereas the standard arithmetic continuum $\mathbb{R}$ of real numbers should be regarded as an arithmetic continuum modulo the Archimedean axiom, Conway's ordered field *No* of surreal numbers may be regarded as a sort of absolute arithmetic continuum modulo NBG. Here we show how the aforementioned formalizations of the theories of du Bois-Reymond, Veronese, and Peirce along with $\mathbb{R}$ are naturally situated in *No*.

Preliminary Remarks on *No*

An ordered class A will be said to be an *absolute linear continuum* if for every pair of subsets X and Y of A where $X < Y$, there is a $y \in A$ such that $X < \{y\} < Y$.

Theorem (Ehrlich 1987) *No* is (up to isomorphism) the unique real-closed field that is an *absolute linear continuum.*

In virtue of the above theorem, *No* not only exhibits all possible types of algebraic and **set**-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, it is to within isomorphism the unique such structure that does. It is ultimately this together with a number of closely related results that underlies our contention that *No* may be naturally regarded as an absolute arithmetic continuum (modulo NBG).



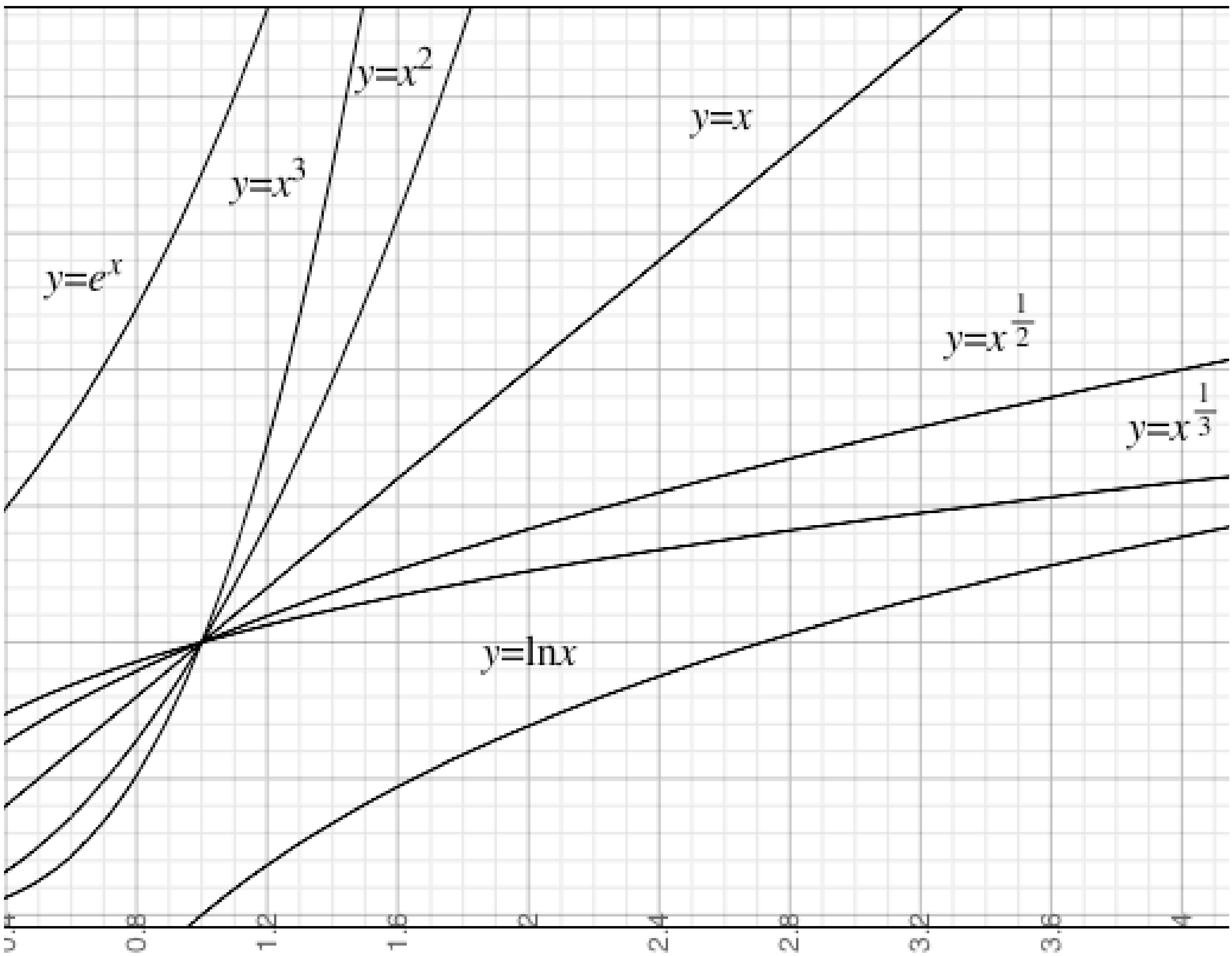
Let $No(\alpha)$ be the subset of No consisting of all surreal numbers having tree rank $< \alpha$.

Theorem (Van den Dries and Ehrlich 2001). $No(\alpha)$ is an ordered field if and only if α is an epsilon number.

Paul du Bois-Reymond's *Infinitärcalcül* (calculus of infinities)

Du Bois-Reymond (1870-1882) erects his calculus primarily on families of increasing functions from $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ to $\mathbb{R}$ such that for each function f of a given family, $\lim_{x\to\infty} f(x) = +\infty$, and for each pair of functions f and g of the family, $0 \le \lim_{x\to\infty} f(x)/g(x) \le +\infty$. He assigns to each such function f a so-called *infinity*, and defines an ordering on the infinities of such functions by stipulating that for each pair of such functions f and g:

$f(x)$ has an infinity *greater than* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = \infty$;
$f(x)$ has an infinity *equal to* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = a \in \mathbb{R}^+$;
$f(x)$ has an infinity *less than* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = 0$.



$y=e^x$
$y=x^3$
$y=x^2$
$y=x$
$y=x^{\frac{1}{2}}$
$y=x^{\frac{1}{3}}$
$y=\ln x$
0.8
1.2
1.6
2
2.4
2.8
3.2
3.6
4

Otto Stolz

"Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes", *Mathematische Annalen* (1883) 22, 504-519.

Stolz considers the set of all functions $f:\mathbb{R}^+ \rightarrow \mathbb{R}^+$ formed by means of finite combinations of the operations +, -, ·, and ÷ from positive rational powers of the functions $x, \ln x, \ln(\ln x), \ldots; e^x, e^{e^x}, e^{e^{e^x}}, \ldots$ where $\ln x$ is the natural logarithm of x and e is the base of the natural logarithm. Following du Bois-Reymond, Stolz assigns to each such function f an *infinity* -- which he denotes by "$\mathfrak{U}(f)$" -- and defines an ordering on the infinities of such functions in the manner specified above. To complete the construction, Stolz defines addition and subtraction of the infinities by the rules:

$$\mathfrak{U}(f) + \mathfrak{U}(g) = \mathfrak{U}(f \cdot g)$$
$$\mathfrak{U}(f) - \mathfrak{U}(g) = \mathfrak{U}(f/g), \text{ if } \mathfrak{U}(f) > \mathfrak{U}(g).$$

Hausdorff (1907), working in a more general setting, calls a maximal set of such functions totally order by the "final order" a *pantachie* and he establishes the following results for an arbitrary pantachie $\mathbb{H}_P$.

Hausdorff (1907). $\mathbb{H}_P$ is an η_1-ordering of power $2^{\aleph_0}$. Moreover, $\mathbb{H}_P$ is (up to isomorphism) the unique η_1-ordering of power $\aleph_1$, assuming the Continuum Hypothesis.

Hausdorff (1909); Boshernitzan (1981). $\mathbb{H}_P$ (with sums and products defined in the manner familiar from the theory of Hardy fields) is a real-closed ordered field.

The Relation Between Hausdorff's Pantachie and *No*.

Let $No(\omega_1)$ be the subset of *No* consisting of all surreal numbers having tree rank $< \omega_1$.

Theorem (Ehrlich). $\mathbb{H}_P$ (considered as an ordered field) is isomorphic with an initial subfield of *No* extending $No(\omega_1)$; assuming the Continuum Hypothesis, $\mathbb{H}_P$ is in fact isomorphic to $No(\omega_1)$. Moreover, the orders of infinity of the members of $\mathbb{H}_P$ is isomorphic to the value group (i.e. the ordered Abelian group of Archimedean classes) of $No(\omega_1)$.

Giuseppe Veronese's non-Archimedean Geometrical Continuum

In his *Fondamenti di geometria* (1891), Giuseppe Veronese introduced the first non-Archimedean geometry and put forth the mathematico-philosophical thesis that the concept of continuity is independent of the Archimedean axiom. To bolster his thesis he introduced an "absolute continuity" condition, which reduces to the Dedekind continuity condition if one assumes the Archimedean axiom, and provided a sketch of a synthetic construction of a non-Archimedean ordered field that is continuous in his sense. His absolute continuity condition, which is now familiar under a host of other names, may be stated as follows:

An ordered field A is continuous (in Veronese's sense) if every *Veronese Cut* of A is determined by an element of A.

A cut (X,Y) of A is a Veronese cut of A if for each positive element a of A there are elements x of X and y of Y such that $y - x < a$.

In a paper, titled *Sui Numeri Transfiniti* (On Transfinite Numbers) (1898), Levi-Civita undertook the complete arithmetization of Veronese's non-Archimedean linear continuum. In the opening paragraphs of the paper Levi-Civita broached these matters thus.

"In a work, published a few years ago, I have shown how by opportune conventions, we can construct a system of finite, infinite and infinitesimal numbers, for which the ordinary rules of calculation are valid. I was led to this system, attempting to develop a fundamental idea of Prof. Veronese in a purely arithmetical fashion. My numbers were not created to represent the entire edifice of Veronese, but only one part of it (being in some respects more general). My research is now complete, showing how it is possible to generate a type of system of symbols that includes the system of Veronese, and for which all of the ordinary laws of arithmetic preserve their validity."

Adopting Levi-Civita's notation, let M be a non-trivial ordered Abelian group written additively. The absolute value of $a \in M$, written $|a|$, is defined as the greatest member of $\{a, -a\}$. Following Levi-Civita, if a and b are distinct members of $M - \{0\}$, then a and b are said to *relatively finite*, if they satisfy the *Archimedean condition*: there is a positive integer k such that $k|b| > |a|$ whenever $|a| > |b|$. On the other hand, and again following Levi-Civita, if $|a| > |b|$, but there is no positive integer k such that $k|b| > |a|$, a is said to be *infinite relative to b*, written $a \cdot\!> b$, and b is said to be *infinitesimal relative to a*.

In modern parlence, elements that are relatively finite in Levi-Civita's sense are said to be

Archimedean equivalent; Archimedean equivalence partitions the elements of $M-\{0\}$ into disjoint classes called *Archimedean classes*. Moreover, as Levi-Civita observes, the Archimedean classes (or *independent systems*, as he calls them) of a non-trivial ordered Abelian group *M* are totally ordered by the condition: an Archimedean class N' succeeds an Archimedean class N'' just in case the members of N' are infinite relative to the members of N'' (1898 in Levi-Civita 1954, p. 322). Employing an obvious abuse of notation, Levi-Civita represents this by writing $N' \cdot\!> N''$.

Now let *M* be a non-trivial ordered Abelian group and for some positive integer *n* let $N^{(n)} \cdot\!> N^{(n-1)} \cdot\!> \ldots \cdot\!> N^{(1)}$ be *n* distinct independent subclasses of *M* ordered by $\cdot\!>$. From each $N^{(i)} \cup \{0\}$ $(1 \le i \le n)$ one selects an *elliptic* subset--a subset such that for each $v \in N^{(i)} \cup \{0\}$ there are at most finitely many members of the subset that are $\ge v$--and the union of the *n* elliptic sets arranged in descending order is said to be a *hyperelliptic set of order n*. A hyperelliptic set of order *n*, thus defined, is a descending ordered set $\nu^{(0)} > \ldots > \nu^{(\alpha)} > \ldots$ indexed over all ordinals less than some nonzero ordinal $\beta \le \omega \cdot n$.

Theorem (Levi-Civita): For each ordered field A and each non-trivial ordered Abelian group M, the set of all formal sums

$$a^{(0)}_{\nu^{(0)}} + \ldots + a^{(\alpha)}_{\nu^{(\alpha)}} + \ldots$$

whose *characteristics* (coefficients) $a^{(0)}, \ldots, a^{(\alpha)}, \ldots$ are in A and whose ordered set $\nu^{(0)} > \ldots > \nu^{(\alpha)} > \ldots$ of *indices* constitutes a hyperelliptic subset of M is an ordered field when order is defined lexicographically, and sums and products are defined termwise in accordance with the conditions

$$a_{\nu^{(\alpha)}} + b_{\nu^{(\alpha)}} = (a+b)_{\nu^{(\alpha)}},$$

$$a_{\nu^{(\alpha)}} \cdot b_{\nu^{(\beta)}} = (a \cdot b)_{\nu^{(\alpha+\beta)}}.$$

$$at^{\alpha} + bt^{\alpha} = (a+b)t^{\alpha} \qquad at^{\alpha} + bt^{\beta} = (a \cdot b)t^{\alpha+\beta}$$

With his construction of *hyperelliptic ordered number fields* (as he calls them) at hand, Levi-Civita turns to the representation of Veronese's non-Archimedean continuum. The representation is carried out in two stages, the first of which consists four parts.

Stage I

To begin with, Levi-Civita lets $V^{(1)}$ be the ordered ring consisting of all finite formal sums

$$a^{(0)}_{\nu^{(0)}} + \ldots + a^{(n)}_{\nu^{(n)}}$$

with integers for characteristics and a descending set of non-negative integers for indices. Following this, he lets $A^{(1)}$ be the hyperelliptic ordered field with characteristics in the reals and indices in $V^{(1)}$, and for each $i > 1$ he lets $A^{(i)}$ be the hyperelliptic ordered field with characteristics in the reals and indices in the additive subgroup $V^{(i)}$ of $A^{(i-1)}$ consisting of those members of the form $a^{(0)}_{\nu^{(0)}} + \ldots + a^{(n)}_{\nu^{(n)}}$ where $a^{(0)}, \ldots, a^{(n)}$ are integers. With the $V^{(i)}$s so defined, he forms "the union V of all the $V^{(i)}$ $(i = 1, 2, \ldots)$". And finally, to complete the first stage of the construction, he forms the hyperelliptic ordered field A' with real characteristics and indices in V.

Stage II

To complete the arithmetization, Levi-Civita supplements A' with all formal sums of the form

$$a_{v^{(0)}}^{(0)} + \ldots + a_{v^{(\alpha)}}^{(\alpha)} + \ldots$$

where

1. α ranges over all ordinals less than a limit ordinal β,
2. every proper truncation of $a_{v^{(0)}}^{(0)} + \ldots + a_{v^{(\alpha)}}^{(\alpha)} + \ldots$ is a member of A',
3. for each $v \in V$, there is an $\alpha < \beta$ such that $v^{(\alpha)} < v$.

In modern parlence, A'' is what order algebrists sometimes call the *Dedekindean completion* of A' and what logicians sometimes call the *Scott completion* of A'.

The Relation Between Levi-Civita's Arithmetization of Veronese's non-Archimedean Continuum and No.

Let $No(\varepsilon_0)$ be the subset of *No* consisting of all surreal numbers having tree rank < (the first epsilon number) ε_0.

Theorem (Ehrlich). A'' is isomorphic with an initial subfield of $No(\varepsilon_0)$

Peircean Linear Continua:
A Proposed Model

a continuum is a collection [of entities]…so vast…that…the continuum is all that is possible, in whatever dimension it be continuous.

Charles Sanders Peirce
The Logic of Relatives (1898)

the possibility of determining more than any given multitude of points…at every part of the line, makes it *continuous*…. [Accordingly,] we define a continuum as that every part of which can be divided into any multitude of parts whatsoever....

Charles Sanders Peirce
Infinitesimals (1900)

We will say that an ordered class A is *absolutely dense* if for every pair of nonempty subsets X and Y of A where $X < Y$, there is a $y \in A$ such that $X < \{y\} < Y$. Moreover, we will say that

An ordered class A is a *Peircean linear continuum* if it is absolutely dense and it contains an isomorphic copy of the ordered set of real numbers that is both cofinal and coinitial with A.

Accordingly, an ordered class A is a Peircean linear continuum if and only if it is absolutely dense and it contains an isomorphic copy of the ordered set of real numbers, say, $A_{\mathbb{R}}$, such that every member of A lies between two members of $A_{\mathbb{R}}$.

Insofar as Peirce appears to have envisioned his linear continuum to be an extension of a Cantor-Dedekind linear continuum, the former of whose "nonstandard" points lie between pairs of members of the Cantor-Dedekind linear continuum in question, a Peircean linear continuum, as defined above, is compatible with this aspect of Peirce's vision.

Now let No_P be the subclass of the ordered class *No* of surreal numbers consisting of all the surreal numbers lying between two of *No*'s real numbers. The relation between No_P and Peircean linear continua is given by

Theorem (Ehrlich): No_P is (up to isomorphism) the unique Peircean linear continuum.

The Peircean Arithmetic Continuum

While Peirce staunchly advocated the use of infinitesimals in the calculus and pictured them manipulated algebraically, as best as we can tell, he never attempted to impose an ordered algebraic structure on his envisioned linear continuum. Be that as it may, we believe that much as it is instructive to compare No_P with *No* from an order-theoretic perspective, it is likewise instructive to compare them from an ordered-algebraic point of view. From this perspective, it is the conception of an *ordered domain* that is fundamental.

Every ordered domain A contains a canonical copy of the ordered domain $\mathbb{Z}$ of integers, namely, the subdomain of all elements of A of the form $n \cdot 1_A$ where n is an integer and 1_A is the unit element of A. Henceforth, we will refer to the elements of this subdomain of A as *the integers of* A. Moreover, an ordered domain A will be said to be *integrally bound* if every element of A lies between two of A's integers. While an integrally bound ordered domain A may contain elements that are infinitesimal, it cannot contain elements that are infinitely large.

It is well known that an ordered field A is real-closed if and only if it satisfies the *intermediate value theorem for polynomials* (*in one variable*) *with coefficients in* A. From a geometrical point of view, this means that if the graph of a polynomial with coefficients in A has points on the opposite sides of a line, then the portion of the graph lying between the two points intersects the given line.

Cherlin and Dickmann (1983) extended this idea to ordered domains, more generally, likewise calling an ordered domain *A* *real-closed* if it satisfies the intermediate value theorem for polynomials (in one variable) with coefficients in *A*.

Theorem (Ehrlich): No_P is (up to isomorphism) the unique integrally bound, real-closed ordered domain that is a Peircean linear continuum.

Theorem (Ehrlich): *No* is No_P's field of fractions.

F. Hausdorff, “Die Graduierung nach dem Endverlauf (Graduation by Final Behavior)”, *Abhandlungen der math. phys. Klasse der königlich sächsischen Gesellschaft der Wissenschaften* vol. 31, 1909, pp. 295-335.

T. Levi-Civita, “Sui Numeri Transfiniti (On Transfinite Numbers)”, *Atti della Reale Accademia dei Lincei, Classe di scienze fisiche, matematiche e naturali, Rendiconti*, *Roma* serie Va, vol. 7, 1898, pp. 91-96, 113-121.

P. Ehrlich, “The Absolute Arithmetic Continuum and its Peircean Counterpart” in *New Essays on Peirce’s Mathematical Philosophy*, edited by Matthew Moore, Open Court Press (forthcoming).