Applications of higher-dimensional forcing

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Motivation

Suppose we want to construct a ccc forcing $\mathbb{P}$ of size ω_1. Then we can proceed as follows:

Let $\langle \sigma_{\alpha\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta \mid \alpha < \beta < \omega_1 \rangle$ be a continuous, commutative system of complete embeddings between forcings.

Let $\mathbb{P}$ be the direct limit of the system and assume that all $\mathbb{P}_\alpha$ are countable. Then every $\mathbb{P}_\alpha$ satifies ccc. Hence by a well-known theorem about finite support iterations $\mathbb{P}$ also satisfies ccc.

Obviously, this only works for $\mathbb{P}$ of size ω_1 because we take a direct limit of countable structures.

What do we have to change to construct a ccc forcing $\mathbb{P}$ of size ω_2 from countable approximations?

1. We need some structure along which we index the approximations and which replaces the ordinal ω_1 with its natural order. An appropriate structure will be given by a simplified $(\omega_1, 1)$-morass. That is, we replace the linear system by a two-dimensional one.

2. We need a replacement for the continuous, commutative system of complete embeddings. We will also construct a continuous, commutative system of embeddings. However, not all of them will be complete.

We have to assume that there exists an $(\omega_1, 1)$-morass to construct a ccc forcing $\mathbb{P}$ of size ω_2. There exists already a very successful method to construct a ccc forcing of size ω_2 assuming only $\Box_{\omega_1}$. This is Todorcevic's method of ordinal walks. So, why is the new method interesting?

1. There exists a variant of it which guarantees that $\mathbb{P}$ has a dense subforcing of size ω_1. Hence $\mathbb{P}$ preserves GCH.

2. It has a natural generalization which allows to construct ccc forcings of size ω_3. Since very little is known about possible structures on ω_3, this might be interesting.

Applications

Theorem

If there exists a simplified $(\omega_1, 1)$-morass, then there is a ccc forcing of size ω_1 that adds an ω_2-Suslin tree.

It was known before that there exists a ccc forcing which adds an ω_2-Suslin tree if $\square_{\omega_1}$ holds (Todorcevic).

Theorem

Assume that there exists a (simplified) $(\omega_1, 1)$-morass. Then there is a ccc forcing which adds a $g : [\omega_2]^2 \longrightarrow \omega$ such that $\{\xi < \alpha \mid g(\xi, \alpha) = g(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.

This was first proved by Todorcevic using only the assumption that $\square_{\omega_1}$ holds. He uses ordinal walks / Δ-functions.

Note that, by the Erdös-Rado theorem, the existence of a function g like in the theorem implies $\neg CH$.

Theorem

There exists consistently a chain $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ has size ω_1 for all $\beta < \alpha < \omega_2$.

This was first proved by Koszmider using ordinal walks.

Let X be a topological space. Its spread is defined by

$$spread(X) = sup\{card(D) \mid D \text{ discrete subspace of } X\}.$$

Theorem (Hajnal,Juhasz - 1967)
If X is a Hausdorff space, then $card(X) \leq 2^{2^{spread(X)}}$.

In his book "Cardinal functions in topology" (1971), Juhasz explicitly asks if the second exponentiation is really necessary. This was answered by Fedorcuk (1975).

Theorem

In L, there exists a 0-dimensional Hausdorff (and hence regular) space with spread ω of size $\omega_2 = 2^{2^{spread(X)}}$.

This is a consequence of $\diamondsuit$ (and GCH).

There was no such example for the case $spread(X) = \omega_1$.
Three-dimensional forcing yields the following:

Theorem

If there is a simplified $(\omega_1, 2)$-morass, then there exists a ccc forcing of size ω_1 which adds a 0-dimensional Hausdorff space X of size ω_3 with spread ω_1.

Hence there exists such a forcing in L. By the usual argument for Cohen forcing, it preserves GCH. So the existence of a 0-dimensional Hausdorff space with spread ω_1 and size $2^{2^{spread(X)}}$ is consistent.

We write

$$\kappa \longrightarrow (\sigma : \tau)^2_\gamma$$

for: Every partition $f : [\kappa]^2 \longrightarrow \gamma$ has a homogeneous set $[A; B] := \{\{\alpha, \beta\} \mid \alpha \in A, \beta \in B\}$ where $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$ and $card(A) = \sigma$, $card(B) = \tau$, i.e. f is constant on $[A; B]$.

We write $\kappa \not\longrightarrow (\sigma : \tau)^2_\gamma$ for the negation of this statement.

Theorem (see above)

If there exitsts an $(\omega_1, 1)$-morass, then there is a ccc forcing for $\omega_2 \not\to (\omega : 2)^2_\omega$.

Question

Assume that there is an $(\omega_1, 2)$-morass. Does then exist a ccc forcing with finite conditions which forces $\omega_3 \not\to (\omega : 2)^2_{\omega_1}$?

Question

Assume we force $\omega_3 \not\to (\omega_2 : 2)^2_{\omega_1}$ with a ccc forcing. Does then $2^\omega = \omega_3$ hold in the generic extension?

Question

What does MA_{ω_3} + *there exists an* $(\omega_1, 2) -$ *morass* imply?

Even though the method generalizes straightforwardly to higher-dimensions, this is not true for the consistency statements. The reason is that the conditions of the forcing have to fit together in more directions, if we go to higher dimensions.

Question Can we find good applications in dimensions higher than two?

Even though the method is inspired by iterated forcing, all my examples use basically finite sets of ordinals as conditions. Hence my examples use rather product forcing than an iteration.

Question Can we find an application which uses a real iteration?

Question Can we do something similar for countable support?

Definition of gap-1 morasses

A simplified $(\kappa, 1)$-morass is a structure $\mathfrak{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa \rangle \rangle$ satisfying the following conditions:

(P0) (a) $\theta_0 = 1$, $\theta_\kappa = \kappa^+$, $\forall \alpha < \kappa \;\; 0 < \theta_\alpha < \kappa$.

(b) $\mathfrak{F}_{\alpha\beta}$ is a set of order-preserving functions $f : \theta_\alpha \to \theta_\beta$.

(P1) $|\mathfrak{F}_{\alpha\beta}| < \kappa$ for all $\alpha < \beta < \kappa$.

(P2) If $\alpha < \beta < \gamma$, then $\mathfrak{F}_{\alpha\gamma} = \{ f \circ g \mid f \in \mathfrak{F}_{\beta\gamma}, g \in \mathfrak{F}_{\alpha\beta} \}$.

(P3) If $\alpha < \kappa$, then $\mathfrak{F}_{\alpha,\alpha+1} = \{id \upharpoonright \theta_\alpha, f_\alpha\}$ where f_α is such that $f_\alpha \upharpoonright \delta = id \upharpoonright \delta$ and $f_\alpha(\delta) \geq \theta_\alpha$ for some $\delta < \theta_\alpha$.

(P4) If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$ and $f_1 \in \mathfrak{F}_{\beta_1\alpha}$, $f_2 \in \mathfrak{F}_{\beta_2\alpha}$, then there are a $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathfrak{F}_{\gamma\alpha}$ and $h_1 \in \mathfrak{F}_{\beta_1\gamma}$, $h_2 \in \mathfrak{F}_{\beta_2\gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(P5) For all $\alpha > 0$, $\theta_\alpha = \bigcup\{f[\theta_\beta] \mid \beta < \alpha, f \in \mathfrak{F}_{\beta\alpha}\}$.

The morass tree

Lemma

Let $\alpha < \beta < \kappa$, $\tau_1, \tau_2 < \theta_\alpha$, $f_1, f_2 \in \mathfrak{F}_{\alpha\beta}$ and $f_1(\tau_1) = f_2(\tau_2)$. Then $\tau_1 = \tau_2$ and $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.

A simplified morass defines a tree $\langle T, \prec \rangle$.

Let $T = \{\langle \alpha, \nu \rangle \mid \alpha \leq \kappa, \nu < \theta_\alpha\}$.
For $t = \langle \alpha, \nu \rangle \in T$ set $\alpha(t) = \alpha$ and $\nu(t) = \nu$.
Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ iff $\alpha < \beta$ and $f(\nu) = \tau$ for some $f \in \mathfrak{F}_{\alpha\beta}$.
If $s \prec t$, then $f \restriction (\nu(s) + 1)$ is uniquely determined by the lemma.
So we may define $\pi_{st} := f \restriction (\nu(s) + 1)$.

The tree maps

Lemma

The following hold:

(a) $\prec$ is a tree, $ht_T(t) = \alpha(t)$.

(b) If $t_0 \prec t_1 \prec t_2$, then $\pi_{t_0 t_1} = \pi_{t_1 t_2} \circ \pi_{t_0 t_1}$.

(c) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $s' \prec t'$ and $\pi_{s' t'} = \pi \upharpoonright (\nu' + 1)$.

(d) Let $\gamma \leq \kappa$, $\gamma \in Lim$. Let $t \in T_\gamma$. Then $\nu(t) + 1 = \bigcup \{rng(\pi_{st}) \mid s \prec t\}$.

Example

As example, we force

$$\omega_2 \not\to (\omega : 2)^2_\omega \qquad .$$

More precisely, we want to add a function $f : [\omega_2]^2 \longrightarrow \omega$ such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.

A forcing

For $a, b \subseteq \omega_2$ set $[a, b] := \{\langle\alpha, \beta\rangle \mid \alpha \in a,\ \beta \in b,\ \beta < \alpha\}$.

Let

$$P := \{p : [a_p, b_p] \to \omega \mid a_p, b_p \subseteq \omega_2 \text{ finite } \}.$$

We set $p \leq q$ iff $q \subseteq p$ and

$$\forall \alpha < \beta \in a_q\ \forall \xi \in (b_p - b_q) \cap \alpha\ p(\alpha, \xi) \neq p(\beta, \xi).$$

Mapping conditions

Let $\pi : \bar{\theta} \longrightarrow \theta$ be an order-preserving map. Then $\pi : \bar{\theta} \longrightarrow \theta$ induces maps $\pi : \bar{\theta}^2 \longrightarrow \theta^2$ and $\pi : \bar{\theta}^2 \times \omega \longrightarrow \theta^2 \times \omega$ in the obvious way:

$$\pi : \bar{\theta}^2 \longrightarrow \theta^2, \quad \langle \gamma, \delta \rangle \mapsto \langle \pi(\gamma), \pi(\delta) \rangle$$

$$\pi : \bar{\theta}^2 \times \omega \longrightarrow \theta^2 \times \omega, \quad \langle x, \epsilon \rangle \mapsto \langle \pi(x), \epsilon \rangle.$$

The recursive definition

Base Case: $\beta = 0$

Then we only need to define $P(1)$.

Let $P(1) := \{p \in P \mid a_p, b_p \subseteq 1\}$.

Successor Case: $\beta = \alpha + 1$

We first define $P(\varphi_\beta)$. Let it be the set of all $p \in P$ such that

(1) $a_p, b_p \subseteq \varphi_\beta$

(2) $f_\alpha^{-1}[p], (id \restriction \varphi_\alpha)^{-1}[p] \in P(\varphi_\alpha)$

(3) $p \restriction ((\varphi_\beta \setminus \varphi_\alpha) \times (\varphi_\alpha \setminus \delta))$ is injective

where f_α and δ are like in (P3) in the definition of a simplified gap-1 morass.

For all $\nu \leq \varphi_\alpha$ $P(\nu)$ is already defined. For $\varphi_\alpha < \nu \leq \varphi_\beta$ set

$$P(\nu) = \{p \in P(\varphi_\beta) \mid a_p, b_p \subseteq \nu\}.$$

Set

$$\sigma_{st} : P(\nu(s)+1) \to P(\nu(t)+1), p \mapsto \pi_{st}[p].$$

Limit Case: $\beta \in Lim$

For $t \in T_\beta$ set $P(\nu(t)+1) = \bigcup\{\sigma_{st}[P(\nu(s)+1)] \mid s \prec t\}$ and $P(\lambda) = \bigcup\{P(\eta) \mid \eta < \lambda\}$ for $\lambda \in Lim$ where

$$\sigma_{st} : P(\nu(s)+1) \to P(\nu(t)+1), p \mapsto \pi_{st}[p].$$