A modular ordinal analysis of subsystems of second order arithmetic below Π^1_3 reflection

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Outline

1. Modular ordinal analysis – Motivation
2. A family of theories T^l suitable for a modular ordinal analysis
3. Sketch of the Main Result
4. Conclusions and future work

Outline

Proof-theoretic ordinal |T| of a theory T of L_2

$\alpha \geq |T|$: bound on depth of cut-free proofs

For all finite sets Γ of L_1 sentences, $T \vdash \Gamma \Longrightarrow PA^* \vdash^{\alpha}_{0} \Gamma$.

$\alpha < |T|$: α is provable in T

$$T \vdash \mathrm{Prog}_{\triangleleft}(U) \rightarrow \alpha \in U.$$

Proof-theoretic ordinal $|\mathsf{T}|$ of a theory T of L_2

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For all finite sets Γ of L_1 sentences, $\mathsf{T} \vdash \Gamma \Longrightarrow \mathsf{PA}^* \vdash^{\alpha}_{0} \Gamma$.

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$$\mathsf{T} \vdash \mathsf{Prog}_{\lhd}(\mathsf{U}) \rightarrow \alpha \in \mathsf{U}.$$

Boundedness Lemma (Pohlers, Beckmann)

$$\mathsf{PA}^* \vdash^{\gamma}_{1} \mathsf{TI}_{\lhd}(\mathsf{U}, \alpha) \Longrightarrow \alpha \leq \gamma$$

(cuts of the form $t \in X$ are still allowed).

Questions answered by an ordinal analysis of T

- What does it cost to eliminate a cut with some axiom A?

$$\overset{*}{\mathsf{T}} + A \mathrel{\vert\!\frac{\alpha}{*}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}} \mathrel{\vert\!\frac{f(\alpha)}{*}} \Gamma.$$

- Lower bounds: How far can we jump?

$$\mathsf{T} \vdash \mathsf{Wo}^{\Pi^1_i}_{\lhd}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}(f(\alpha))?$$

$$\mathsf{Wo}^{\Pi^1_i}_{\lhd}(\alpha) := \forall X \forall e [\mathsf{Prog}_{\lhd}(\{\beta : \pi^1_i(X, \beta, e)\}) \rightarrow \pi^1_i(X, \alpha, e)].$$

Descriptions of T

Let T be a subsystem of second order arithmetic. A representable normal function f : Ord $\rightarrow$ Ord describes a formal theory T, if

1. $\overset{*}{\mathsf{T}}[\overset{*}{\mathsf{S}}] \mathrel{\vert\!\frac{\alpha}{*}} \Gamma \Longrightarrow \mathsf{PA}^*[\overset{*}{\mathsf{S}}] \mathrel{\vert\!\frac{f(\alpha)}{*}} \Gamma$, and
2. $\mathsf{T} \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}(f(\alpha))$ (Π^1_{i+1} is the formula complexity of T).

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Example: The normal function $\alpha \mapsto \varphi 1 \alpha$ describes ACA_0

- If $\mathrm{ACA}_0^*[\overset{*}{\mathrm{S}}] \overset{\alpha}{\underset{*}{\vdash}} \Gamma$, then $\mathrm{PA}^*[\overset{*}{\mathrm{S}}] \overset{\alpha}{\underset{\omega^1}{\vdash}} \Gamma$. Hence $\mathrm{PA}^*[\overset{*}{\mathrm{S}}] \overset{\varphi 1 \alpha}{\underset{*}{\vdash}} \Gamma$.
- (ACA) is $\forall X \forall e \exists Y [Y = (\mathcal{J}^X)_e]$, where $\mathcal{J}^X := \{\langle x, e\rangle : \pi^0_1(X, x, e)\}$, i.e. ACA_0 is Π^1_2.
- The class $\mathcal{Z} := \{\gamma : \mathrm{Wo}_{\triangleleft}(\varphi 1 \gamma)\}$ is progressive. $\mathcal{Z}$ is Π^1_1.
- Hence $\mathrm{ACA}_0 \vdash \mathrm{Wo}_{\triangleleft}^{\Pi^1_1}(\alpha) \to \mathrm{Wo}_{\triangleleft}(\varphi 1 \alpha)$.

Modular ordinal analysis

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We introduce a family of theories with the following properties:

If f_S and f_T describe S and T, then

- $f_\mathsf{T} \circ f_\mathsf{S}$ describes $\mathsf{S}[\mathsf{T}]$,
- f'_T describes $\mathsf{lim}(\mathsf{T})$,
- if f_T is $\alpha \mapsto \varphi^{k+1}\vec{\gamma}0\alpha\vec{0}$, then $\alpha \mapsto \varphi^{k+1}\vec{\gamma}\alpha0\vec{0}$ describes Π^1_2-Refl_T.

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Approximate limits by over operations.
Approximate Π^1_2-Refl_T by limit operations and Π^1_2-$\mathsf{Refl}_{\mathsf{T}'}$, where T' is "simpler" than T.

Towards Π^1_3-reflection

- $\mathsf{S}^0 := \Pi^0_1\text{-}\mathsf{CA}^-_0$ and $\mathsf{S}^{n+1} := \Pi^0_1\text{-}\mathsf{CA}^-_0 + \Pi^1_2\text{-}\mathsf{Refl}_{\mathsf{S}^n}$,
- $\mathsf{S} := \Pi^0_1\text{-}\mathsf{CA}^-_0 + \Pi^1_3\text{-}\mathsf{Refl}_{\mathsf{T}^0}$.

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Remark (Π^1_2-$\mathsf{Refl}_{\mathsf{S}^k}$)

Suppose that Γ is a finite set of Σ^1_2 formulas. Then, $\neg \bigvee \Gamma$ is Π^1_2, and

$$\neg \bigvee \Gamma \quad \to \quad \exists M[M \models \mathsf{S}^k \wedge \neg \bigvee \Gamma \restriction M], \text{ i.e.}$$

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Lemma

Let Γ be a finite set of Σ^1_2 formulas. Then

$$\mathsf{S} \mathrel{\vert\!\frac{k}{*}} \Gamma \Longrightarrow \mathsf{S}^k \vdash \Gamma.$$

Proof (by induction on the depth k of the derivation)

- S $\overset{k+1}{\underset{*}{\vdash}}$ Γ is obtained by a cut with an instance A of Π^1_3-reflection.

  Assume we reflected the Π^1_3 formula $\forall X \exists Y B(X,Y)$.

  So S $\overset{k+1}{\underset{*}{\vdash}}$ $\Gamma, \neg A$. Now $\wedge$- and $\forall$-inversion and the I.H. yield

$$\begin{aligned} \mathsf{S}^k &\vdash \Gamma, \exists Y B(U,Y), \\ \mathsf{S}^k &\vdash \Gamma, M \not\models \mathsf{S}^0, \neg(\forall X \exists Y B(X,Y) {\restriction} M). \end{aligned}$$

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\mathsf{S}^{k+1} &\vdash M \not\models \mathsf{S}^k, \Gamma {\restriction} M, \forall X \exists Y B(X,Y) {\restriction} M, \\
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- A cut yields $\mathsf{S}^{k+1} \vdash \Gamma, \Gamma \restriction M, M \not\models \mathsf{S}^k$.

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- A cut yields $\mathsf{S}^{k+1} \vdash \Gamma, \Gamma{\upharpoonright}M, M \not\models \mathsf{S}^k$.
- A cut with $\mathsf{S}^{k+1} \vdash \Gamma, \exists M[M \models \mathsf{S}^k \wedge \neg \bigvee \Gamma{\upharpoonright}M]$ yields $\mathsf{S}^{k+1} \vdash \Gamma$.

Approximating Π^1_2-reflection by iterating limits

Let $\mathsf{T}^0 := \Pi^0_1\text{-}\mathsf{CA}_0^-$, $\mathsf{T}^\Omega := \Pi^1_2\text{-}\mathsf{Refl}_{T^0}$ and $\mathsf{T}^{n+1} := \lim(\mathsf{T}^n)$.

Lemma

$$\mathsf{T}^\Omega \mathrel{\mathop{\vdash}^{n}_{*}} \Gamma \Longrightarrow \mathsf{T}^n \mathrel{\vdash_{*}} \Gamma.$$

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Lemma (as expected to hold for the semi-formal system)

$$\overset{*}{\mathsf{T}}{}^\Omega \mathrel{\frac{\alpha}{*}\!\!|} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}{}^\alpha \mathrel{\frac{\alpha}{*}\!\!|} \Gamma.$$

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Remark: T^α can be axiomatized by a Π^1_2 sentence.

- In T^Ω, we can reflect the Π^1_2 sentence axiomatizing T^γ.
- $\mathsf{T}^\Omega \vdash \mathsf{Prog}_\lhd\{\beta : \forall Z \exists X[Z \dot\in X \wedge X \models \mathsf{T}^\beta\}$ (T^Ω entails $(\Sigma^1_1\text{-DC})$),
- Hence $\mathsf{T}^\Omega \vdash \mathsf{Wo}_\lhd^{\Pi^1_2}(\alpha) \to \mathsf{T}^\alpha$.

A family T^l of theories

- $\mathsf{T}^0 := \Pi^0_1\text{-}\mathsf{CA}^-_0$,
- $\mathsf{T}^\Omega := \Pi^1_2\text{-}\mathsf{Refl}_{\mathsf{T}^0}$,
- $\mathsf{T}^{\vec{\gamma},\Omega,0,\vec{0}} := \Pi^1_2\text{-}\mathsf{Refl}_{\mathsf{T}^{\gamma,0,\Omega,\vec{0}}}$.

- $\mathsf{T}^{1,0} := \lim(\mathsf{T}^\Omega)$,
- $\mathsf{T}^{\vec{\gamma},1,\vec{0},0} := \lim(\mathsf{T}^{\vec{\gamma},\Omega,\vec{0}})$,
- $\mathsf{T}^{\vec{\gamma},\alpha+1} := \lim(\mathsf{T}^{\vec{\gamma},\alpha})$.

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In essence, we have the following:

Let $l(\mathsf{T})$ denote $\lim(\mathsf{T})$ and $\mathsf{p}(\mathsf{T})$ denote $\Pi^1_2\text{-}\mathsf{Refl}_\mathsf{T}$.

- $\mathsf{T}^{\Omega,0,\ldots,0} := p^{k-1}\ (\mathsf{T}^0)$,
- $\mathsf{T}^{1,0,0,\ldots,0} := (l \circ p^{k-1})^1\ (\mathsf{T}^0)$.

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In essence, we have the following:

Let $l(\mathsf{T})$ denote $\lim(\mathsf{T})$ and $\mathsf{p}(\mathsf{T})$ denote Π^1_2-Refl_T.

- $\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_0} := l^{\alpha_0} \circ (l \circ p^1)^{\alpha_1} \circ \ldots \circ (l \circ p^{k-1})^{\alpha_{k-1}} \ (\mathsf{T}^0)$, and
- $\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_{i+1},\Omega,0,\ldots,0} := p^{i+1} \circ (l \circ p^{i+1})^{\alpha_{i+1}} \circ \ldots \circ (l \circ p^{k-1})^{\alpha_{k-1}} \ (\mathsf{T}^0)$.

Theorem

- $|\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_0}| = \varphi^{k+1}\alpha_{k-1}\ldots\alpha_0 0$,
- $|\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_{i+1},\Omega,0,\ldots,0}| = \varphi^{k+1}\alpha_{k-1}\ldots\alpha_{i+1}\omega 0\ldots 0$.

Outline

Fixing notation

- L_2: a standard language of second order arithmetic with a free relation symbol U.
- L: as L_2, but with set terms $S ::= X|(S)_t$. $\quad ((S)_t = \{x : \langle x, t\rangle \in S\})$
- $L(M_0, \ldots, M_{k-1})$ extends L by set constants M_i ($0 \leq i < k$). In this situation it is convenient to define $M_{-1} := \emptyset$ and $M_k := \mathcal{V}$.
- L^*: formulas of L^* do not contain free number variables. They may contain free set variables.

Abbreviations

- $S = T := \forall x[x \in S \leftrightarrow x \in T]$,
- $S \dot{\in} T := \exists x[S = (T)_x]$, (S over T)
- $S \dot{\subseteq} T := \forall x \exists y[(S)_x = (T)_y]$,
- $S \dot{=} T := S \dot{\subseteq} T \wedge T \dot{\subseteq} S$,
- $\text{trans}(S) := \forall x, y \exists z[(X)_{x,y} = (X)_z]$.

Fixing notation

The $X \models A$ formula

- For an L formula $A(U)$, $(\mathcal{Q}X \dot{\in} S)A[X/U] := \mathcal{Q}xA[(S)_x/U]$.
- $A{\restriction}S$ is obtained from A by relativizing all set quantifiers to S.
- $S \models A$ is $A{\restriction}S$ where all free set variables U_i are substituted by $(S)_{u_i}$. Assume that no conflicts arise.
- $\Gamma{\restriction}S$ is $\{B{\restriction}S : B \in \Gamma\}$ and $S \models \Gamma$ is $\{S \models B : B \in \Gamma\}$.

Remark

$$\begin{array}{ccc}
\exists YA(U,Y) & \xrightarrow{C \mapsto M \models C} & \exists yA((S)_u,(S)_y) \\
\big\downarrow{\scriptstyle \forall X} & & \big\downarrow{\scriptstyle \forall x} \\
\forall X\exists YA(X,Y) & \xrightarrow{C \mapsto M \models C} & \forall x\exists yA((S)_x,(S)_y)
\end{array}$$

The theory BASE

- Work in L or $L(M_0, \ldots, M_{k-1})$. As L_2 but with set terms $S ::= M_i | X | (S)_t$. It is convenient to define $M_{-1} := \emptyset$ and $M_k := \mathcal{V}$.
- Basic axioms: $\Gamma, A, \sim A$, for each atom A.
- Propositional rules and quantifier rules.
- Cut rules:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

  The formulas A and $\neg A$ are the *cut formulas* of the cut.
- Set term rules: For each number terms s and t and each set term S,

$$\frac{\Gamma, \langle s, t \rangle \in S}{\Gamma, s \in (S)_t} \quad \frac{\Gamma, \langle s, t \rangle \notin S}{\Gamma, s \notin (S)_t}$$

The theory BASE*

- L*: L without free number variables.
- BASE* is a semi-formal Tail-style system. All rules apply only to finite sets of formulas of L*.
- Basic axioms of BASE*: For all true literals A, all atoms $B(\vec{u})$ and number terms $\vec{s}, \vec{t}$ with $\vec{s}^{\mathbb{N}} = \vec{t}^{\mathbb{N}}$:

$$\Gamma, A \qquad \text{and} \qquad \Gamma, \sim B(\vec{s}), B(\vec{t})$$

- ω-rule:

$$\frac{\Gamma, A(t) \text{ for all closed number terms } t}{\Gamma, \forall x A(x)} (\omega\text{-rule}).$$

$\mathsf{T} \overset{\alpha}{\underset{\mathcal{C}}{\vdash}} \Gamma$, $\mathsf{T} \underset{*}{\vdash} \Gamma$

Definition ($\mathsf{T} \overset{\alpha}{\underset{\mathcal{C}}{\vdash}} \Gamma$)

$\mathcal{C}$ is a set of L or L^* formulas.

- If Γ is an axiom of T, then $\mathsf{T} \overset{\alpha}{\underset{\mathcal{C}}{\vdash}} \Gamma$ for all ordinals α and ρ.
- If $\mathsf{T} \overset{\alpha_i}{\underset{\mathcal{C}}{\vdash}} \Gamma_i$ and $\alpha_i < \alpha$ hold for all premises Γ_i of a rule that is not a cut, or whose cut-formulas $A, \neg A$ are elements of the set $\mathcal{C}$, then $\mathsf{T} \overset{\alpha}{\underset{\mathcal{C}}{\vdash}} \Gamma$ holds for the conclusion of this rule.

Definition ($\mathsf{T} \underset{*}{\vdash} \Gamma$)

Let T be a theory extending BASE or BASE^*. Then $\mathsf{T} \underset{*}{\vdash} \Gamma$ is short for $\mathsf{T} \underset{\mathsf{T}(*)}{\vdash} \Gamma$, where $\mathsf{T}(*)$ is closed under negation and contains each formula that is a main formula in the conclusion of an instance of a non-logical axiom or rule of T.

The theories T^l are indexed with labels l

Definition (Lables l_k^Ω)

A k-label is a sequence of length k of ordinals from $\Omega \cup \{\Omega\}$ that has one of the following forms:

- ϵ is the empty label,
- $l = (\alpha_{k-1}, \ldots, \alpha_0)$, with $\alpha_i \in \Omega$,
- $l = (\alpha_{k-1}, \ldots, \alpha_i, \Omega, 0, \ldots, 0)$, with $\alpha_i \in \Omega$

Lim is the set of labels containing an Ω. We let Λ range over Lim, the so-called hard limit labels.

Labels cont.

- Let $<$ be the lexicographic ordering on k-labels. $l+1$ $(\Lambda+1)$ is the $<$-successor on k-labels. E.g. $(1, \Omega, 0) + 1 = (2, 0, 0)$.

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- If $\Lambda = (\alpha_{k-1}, \ldots, \alpha_i, \Omega, 0, \ldots, 0)$, then

  $\Lambda^- := (\alpha_{k-1}, \ldots, \alpha_i, 0, \Omega, 0, \ldots, 0)$, and

  $\Lambda[\alpha] := (\alpha_{k-1}, \ldots, \alpha_i, \alpha, 0, 0, \ldots, 0)$.

Labels cont.

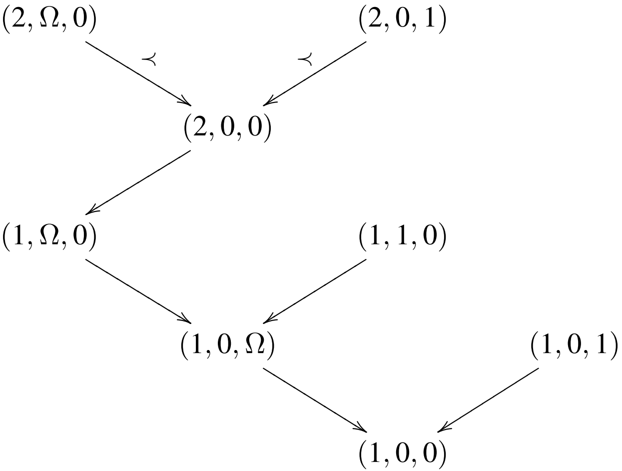
- Let $<$ be the lexicographic ordering on k-labels. $l+1$ ($\Lambda+1$) is the $<$-successor on k-labels. E.g. $(1,\Omega,0)+1=(2,0,0)$.
- If $\Lambda=(\alpha_{k-1},\ldots,\alpha_i,\Omega,0,\ldots,0)$, then

  $\Lambda^- := (\alpha_{k-1},\ldots,\alpha_i,0,\Omega,0,\ldots,0)$, and

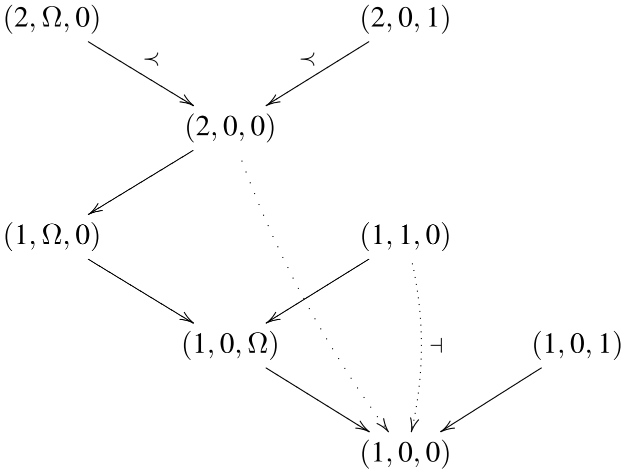
  $\Lambda[\alpha] := (\alpha_{k-1},\ldots,\alpha_i,\alpha,0,0,\ldots,0)$.
- $\mathsf{tail}(\Lambda) := \Lambda_{\mathsf{tail}} := (\Omega,0,\ldots,0)\in \mathsf{I}_i^\Omega$ and

  $\mathsf{head}(\Lambda) := \Lambda_{\mathsf{head}} := (\alpha_{k-1},\ldots,\alpha_i,\vec{0})\in \mathsf{I}_k^\Omega$.

The orderings $\prec$ and $\dashv$ illustrated



The orderings $\prec$ and $\dashv$ illustrated



Composite labels $c, d \in \mathcal{F}$

Instead of $\mathsf{T}^{l_1}[\mathsf{T}^{l_2}]$, we write T^{l_1,l_2}. To generalize, we call finite sequences of labels composite labels.

- The set of all composite labels of length k is $\mathcal{F}_k$.
- If $c = (l_{k-1}, \ldots, l_0)$, $d = (l'_{k'-1}, \ldots, l'_0)$ and $l \in \mathsf{I}^\Omega$, then $(l, c) := (l, l_{k-1}, \ldots, l_0) \in \mathcal{F}_k$ and $(c, d) := (l_{k-1}, \ldots, l_0, l'_{k'-1}, \ldots, l'_0) \in \mathcal{F}_{k+k'}$.

The axioms $\breve{\mathsf{T}}^l$ of the theories T^l ($l \in \mathsf{I}^\Omega$)

$M(U, V, l)$ is a Σ^1_1 formula expressing that U is a transitive model of T^l above Z. Each theory T^l extends BASE by set term induction, $\forall X \mathsf{TI}_{\prec}(X, l)$ and the universal closure of

- $\breve{\mathsf{T}}^\epsilon := \top$,
- $\breve{\mathsf{T}}^0 := \exists X((\mathcal{J}^{\mathsf{P}})_e = X)$,
- $\breve{\mathsf{T}}^{l+1} := \lim(\mathsf{T}^l) := \exists X\, M(X, Z, l)$,
- $\breve{\mathsf{T}}^\lambda := (\forall l \prec \lambda)\exists X\, M(X, Z, l)$, $\quad (\neq \bigcup_{l \prec \lambda} \exists X\, M(X, Z, l)!)$
- $\breve{\mathsf{T}}^\Lambda := \Pi^1_2\text{-Refl}_{\mathsf{T}^{\Lambda^-}} =$

$$\Pi^1_2(Z, e) \rightarrow \exists X[M(X, Z, \Lambda^-) \wedge X \models \Pi^1_2(Z, e)].$$

P serves as a placeholder and is later replaced by a set term.

The theories T^c $(c \in \mathcal{F}_{k+1})$

For each composite label $c \in \mathcal{F}_k^{\Phi_0}$ and each label $l \in \mathsf{I}^\Omega$, the theory $\mathsf{T}^{l,c}$ extends base by

- $\mathsf{M}_{k-1} \models \mathsf{T}^c$,
- $\breve{\mathsf{T}}^l[\mathsf{M}_{k-1}/\mathsf{P}]$, (recall: $\mathsf{M}_{-1} := \emptyset$)
- $\forall X \mathsf{TI}_{\prec}(X, l)$,
- $\mathsf{M}_{i-1} \dot{\in} \mathsf{M}_i \quad (0 \leq i < k-1)$.

The theories T^c $(c \in \mathcal{F}_{k+1})$

For each composite label $c \in \mathcal{F}_k^{\Phi_0}$ and each label $l \in \mathsf{I}^\Omega$, the theory $\mathsf{T}^{l,c}$ extends base by

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- $\breve{\mathsf{T}}^l[\mathsf{M}_{k-1}/\mathsf{P}]$, (recall: $\mathsf{M}_{-1} := \emptyset$)
- $\forall X \mathsf{TI}_{\prec}(X, l)$,
- $\mathsf{M}_{i-1} \dot{\in} \mathsf{M}_i \quad (0 \leq i < k-1)$.

Lemma

- *if* $d \in \mathcal{F}_k$ *and* $\mathsf{T}^c \vdash \Gamma$, *then* $\mathsf{T}^{c,d} \vdash \Gamma[\mathsf{M}_{i+k}/\mathsf{M}_i]$,
- *if* $c \in \mathcal{F}_{k'}$ *and* $\mathsf{T}^d \vdash \Gamma$, *then* $\mathsf{T}^{c,d} \vdash \Gamma \restriction \mathsf{M}_{k'-1}$,
- *if* $l' \dashv l$, *then* $\mathsf{T}^l \vdash \breve{\mathsf{T}}^{l'}$, *i.e.* $\mathsf{BASE} \vdash X \models \mathsf{T}^l \rightarrow X \models \mathsf{T}^{l'}$.

Working in T^l: A first example

$$\mathsf{CHSeq}^{\mathsf{T}^l}(F, Z, \beta) := (\forall \beta \lhd \alpha)[M((F)_\beta, (F)_{\lhd\beta}, l) \land Z \dot{\in} (F)_\beta].$$

Aim: For each $\gamma < \omega^\alpha$, $\mathsf{T}^{1,\alpha} \vdash \forall Z \exists F \mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, Z, \gamma)$.

Lemma

The following is provable in $\mathsf{T}^{1,\alpha}$*:*

$$\forall F \exists G[\mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, Z, \beta) \to \mathsf{CHSeq}^{\mathsf{T}^\Omega}(G, Z, \beta + \omega^\alpha) \land (F)_{\lhd\beta} = (G)_{\lhd\beta}].$$

A first example, cont.

Lemma

The following is provable in $\mathsf{T}^{1,\alpha}$*:*

$$\forall F \exists G[\mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, Z, \beta) \to \mathsf{CHSeq}^{\mathsf{T}^\Omega}(G, Z, \beta+\omega^\alpha) \wedge (F)_{\lhd\beta} = (G)_{\lhd\beta}].$$

Proof: The step from $\alpha+1$ to $\alpha+2$

- Say $\mathsf{CHSeq}^{\mathsf{T}^\Omega}(F_0, \beta_0)$. Pick $X \models \mathsf{T}^{1,\alpha+1}$ with $F_0 \dot\in X$.
- $(\forall F \dot\in X)(\exists G \dot\in X)[\mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, \beta) \to \mathsf{CHSeq}^{\mathsf{T}^\Omega}(G, \beta + \omega^{\alpha+1})]$.
- Pick an Y with $Y \models \mathsf{T}^\Omega$ that contains X.
- $Y \models \forall F \exists G[F \dot\in X \wedge \mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, \beta) \to G \dot\in X \wedge \mathsf{CHSeq}^{\mathsf{T}^\Omega}(G, \beta+\omega^{\alpha+1})]$.
- Since $Y \models (\Sigma^1_1\text{-DC})$, we get G with $\mathsf{CHSeq}^{\mathsf{T}^\Omega}(G, \beta_0 + \omega^{\alpha+2})$.

Outline

1. Modular ordinal analysis – Motivation
2. A family of theories T^l suitable for a modular ordinal analysis
3. Sketch of the Main Result
4. Conclusions and future work

Descriptions of T

Definition (Description from above)

$f : \mathrm{Ord} \to \mathrm{Ord}$ describes T^c from above, if for any T^d ($d \in \mathcal{F}_k$) and each finite set of elementary formulas Γ of $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-1})$,

$$\overset{*}{\mathsf{T}}{}^{c,d} \overset{<\alpha}{\underset{*}{\vdash}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}{}^{\epsilon,d} \overset{<f(\alpha)}{\underset{*}{\vdash}} \Gamma.$$

Definition (Description from below)

A representable normal $f : \mathrm{Ord} \to \mathrm{Ord}$ describes T^c from below, if

$$\mathsf{T}^c \vdash \mathsf{Wo}^{\Pi^1_i}_{\lhd}(\alpha) \to \mathsf{Wo}^{\mathsf{M}_0}_{\lhd}(f(\alpha)).$$

where $i = 2$ if $c(k-1)$ is a hard limit and $i = 1$ otherwise.

$f : \mathrm{Ord} \to \mathrm{Ord}$ describes T^c if it describes T^c from above and from below.

The main result

If $l \in \mathsf{I}_k^{\Omega}$, then $\varphi^{k+1} l\alpha := \varphi^{k+1} l(k-1) \dots l(0)\alpha$.

Theorem (Main Theorem)

For each k-label $0 < l$ that is not a hard limit, and each hard limit Λ,

$$\alpha \mapsto \varphi^{k+1} l\alpha \text{ describes } \mathsf{T}^l \quad \text{and} \quad \alpha \mapsto \varphi^{k+1}\Lambda[\alpha]0 \text{ describes } \mathsf{T}^{\Lambda}.$$

Corollary

- $|\mathsf{T}^{l+1}| = \varphi^{k+1}(l+1)0$,
- $|\mathsf{T}^{\lambda}| = \varphi^{k+1}\lambda\omega$, *(cf. $|\bigcup_{l \dashv \lambda} \mathsf{T}^l| = \varphi^{k+1}\lambda 0$).*
- $|\mathsf{T}^{\Lambda}| = |\mathsf{T}^{\Lambda[\omega]}| = \varphi^{k+1}\Lambda[\omega]0$.

The main result

Corollary

- $|\mathsf{T}^{l+1}| = \varphi^{k+1}(l+1)0$,
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- $|\mathsf{T}^{\Lambda}| = |\mathsf{T}^{\Lambda[\omega]}| = \varphi^{k+1}\Lambda[\omega]0$.

The main result

Corollary

- $|\mathsf{T}^{l+1}| = \varphi^{k+1}(l+1)0$,
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- $|\mathsf{T}^{\Lambda}| = |\mathsf{T}^{\Lambda[\omega]}| = \varphi^{k+1}\Lambda[\omega]0$.

Proof.

- $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_2}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[\alpha]0)$, hence $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[n]0)$, for each $n \in \mathbb{N}$.

□

The main result

Corollary

- $|\mathsf{T}^{l+1}| = \varphi^{k+1}(l+1)0$,
- $|\mathsf{T}^{\lambda}| = \varphi^{k+1}\lambda\omega$, *(cf.* $|\bigcup_{l \dashv \lambda} \mathsf{T}^l| = \varphi^{k+1}\lambda 0$*)*.
- $|\mathsf{T}^{\Lambda}| = |\mathsf{T}^{\Lambda[\omega]}| = \varphi^{k+1}\Lambda[\omega]0$.

Proof.

- $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_2}(\alpha) \to \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[\alpha]0)$, hence $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[n]0)$, for each $n \in \mathbb{N}$.
- If $\mathsf{T}^{\Lambda} \mathrel{\mathop{\vdash}\limits^{n}_{*}} \Gamma$, then $\mathsf{T}^{\Lambda[n]} \vdash \Gamma$. So $\overset{*}{\mathsf{T}}{}^{\Lambda[n]} \mathrel{\mathop{\vdash}\limits^{<\omega^2}_{*}} \Gamma$, i.e. $\overset{*}{\mathsf{T}}{}^{\epsilon} \mathrel{\mathop{\vdash}\limits^{<\varphi^{k+1}\Lambda[n]\omega^2}_{*}} \Gamma$.

□

Some facts on ordinals

Let $O \subseteq \mathrm{Ord}$ and $f : \mathrm{Ord} \to \mathrm{Ord}$ be a normal function.

- If $O \subseteq \mathrm{Ord}$, then $\mathrm{Ord}_O : O \to \mathrm{Ord}$ is the unique order isomorphism.

  Thus, $\mathrm{Ord}_O(\alpha)$ is the αth element of O, which we also denote by $O(\alpha)$.

- $\mathrm{fix}(f) := \{\alpha : \alpha = f(\alpha)\}$ and $f' : \mathrm{Ord} \to \mathrm{fix}(f)$ enumerates these fixed points, i.e.

  $f'(\alpha) := \mathrm{fix}(f)(\alpha)$.

A hierarchy of normal functions g_l : Ord $\to$ Ord

Definition (The functions g_l)

For $l \in \mathsf{I}^{\Omega}$, we define a normal function $g_l : \Omega \to \Omega$ by recursion on $\prec$:
$g_\epsilon :=$ id$_\Omega$, the identity on Ω, and

1. $g_0 := \alpha \mapsto \omega^\alpha$,
2. $g_{l+1} := g_l'$,
3. $g_\lambda := \alpha \mapsto O(\alpha)$, for $O := \bigcap\{\text{fix}(g_l) : l \dashv \lambda\}$,
4. $g_\Lambda := \alpha \mapsto g_{\Lambda[\alpha]}(0)$.

Lemma

1. *If $l \in \mathsf{I}^{\Omega}_k$ is not a hard limit, then $g_l(\alpha) = \varphi^{k+1} l \alpha$.*
2. *If $\Lambda \in \mathsf{I}^{\Omega}_k$ is a hard limit, then $g_\Lambda(\alpha) = \varphi^{k+1} \Lambda[\alpha] 0$.*

The operation S[T]

Lemma

Let $c \in \mathcal{F}_{k'}$ and $d \in \mathcal{F}_k$ be composite labels. Further assume that f and g describe T^c and T^d from above. Then $g \circ f$ describes $\mathsf{T}^{c,d}$ from above.

Proof.

- If $\mathsf{T}^{c,d,e} \frac{\alpha}{*} \Gamma$, then $\mathsf{T}^{\epsilon,c,d} \frac{f(\alpha)}{*} \Gamma$.
- Then $\mathsf{T}^{d,e} \frac{f(\alpha)}{*} \Gamma$ (Lifting Lemma).
- Hence $\mathsf{T}^{\epsilon,e} \frac{g \circ f(\alpha)}{*} \Gamma$.

□

The operation S[T]

Lemma

Let $c \in \mathcal{F}_{k'}$ and $d \in \mathcal{F}_k$ be composite labels. Further assume that f and g describe T^c and T^d from below. Then $g \circ f$ describes $\mathsf{T}^{c,d}$ from below.

Proof.

- $\mathsf{T}^c \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(f(\alpha))$, hence

  $\mathsf{T}^{c,d} \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_k}(f(\alpha))$.

□

The operation S[T]

Lemma

Let $c \in \mathcal{F}_{k'}$ and $d \in \mathcal{F}_k$ be composite labels. Further assume that f and g describe T^c and T^d from below. Then $g \circ f$ describes $\mathsf{T}^{c,d}$ from below.

Proof.

- $\mathsf{T}^c \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \to \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(f(\alpha))$, hence

  $\mathsf{T}^{c,d} \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \to \mathsf{Wo}_{\lhd}^{\mathsf{M}_k}(f(\alpha))$.

- $\mathsf{T}^d \vdash \mathsf{Wo}_{\lhd}^{\Pi^1_i}(\alpha) \to \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(g(\alpha))$, hence

  $\mathsf{T}^{c,d} \vdash \mathsf{Wo}_{\lhd}^{\Pi^0_i(\mathsf{M}_{k-1})}(\alpha) \to \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(g(\alpha))$.

□

The operation S[T]

Lemma

Let $c \in \mathcal{F}_{k'}$ and $d \in \mathcal{F}_k$ be composite labels. Further assume that f and g describe T^c and T^d from below. Then $g \circ f$ describes $\mathsf{T}^{c,d}$ from below.

Proof.

- $\mathsf{T}^c \vdash \mathsf{Wo}_{\lhd}^{\Pi_i^1}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(f(\alpha))$, hence

  $\mathsf{T}^{c,d} \vdash \mathsf{Wo}_{\lhd}^{\Pi_i^1}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_k}(f(\alpha))$.
- $\mathsf{T}^d \vdash \mathsf{Wo}_{\lhd}^{\Pi_i^1}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(g(\alpha))$, hence

  $\mathsf{T}^{c,d} \vdash \mathsf{Wo}_{\lhd}^{\Pi_i^0(\mathsf{M}_{k-1})}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}^{\mathsf{M}_0}(g(\alpha))$.
- Since $\Pi_i^0(\mathsf{M}_{k-1}) \dot{\subseteq} \mathsf{M}_k$, we have $\mathsf{Wo}_{\lhd}^{\mathsf{M}_k}(f(\alpha)) \rightarrow \mathsf{Wo}_{\lhd}^{\Pi_i^0(\mathsf{M}_{k-1})}$. The claim follows.

□

Lifting Lemma and Induction Lemma

Lemma (Lifting Lemma)

Let $f \in \mathcal{F}_k$ *and* Γ *a finite set of elementary* $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-2})$ *formulas, then*

$$\overset{*}{\mathsf{T}}{}^{\epsilon,f} \mathrel{\substack{\alpha \\ \vdash \\ *}} \Gamma{\restriction}\mathsf{M}_{k-1} \iff \overset{*}{\mathsf{T}}{}^{f} \mathrel{\substack{\alpha \\ \vdash \\ *}} \Gamma.$$

Lifting Lemma and Induction Lemma

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Let $f \in \mathcal{F}_k$ *and* Γ *a finite set of elementary* $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-2})$ *formulas, then*

$$\overset{*}{\mathsf{T}}{}^{\epsilon,f} \mathrel{\mid\!\frac{\alpha}{*}} \Gamma{\restriction}\mathsf{M}_{k-1} \iff \overset{*}{\mathsf{T}}{}^{f} \mathrel{\mid\!\frac{\alpha}{*}} \Gamma.$$

Lemma (Induction Lemma)

Let $f \in \mathcal{F}_k$ *and* Γ *a finite set of elementary* $\mathsf{L}(\mathsf{M}_0, \ldots, \mathsf{M}_{k-2})$ *formulas, then*

$$\mathsf{T}^{1,f} \vdash \Gamma{\restriction}\mathsf{M}_{k-1} \iff \mathsf{T}^{f} + (\textit{Full Induction}) \vdash \Gamma.$$

From T^l to T^{l+1} – the limit of a theory

Lemma

If f describes T^l from below, then f' describes T^{l+1} from below.

Proof.

- to show: $\mathsf{T}^{l+1} \to \mathsf{Wo}_{\lhd}^{\Pi^1_1}(\alpha) \to \mathsf{Wo}_{\lhd}(f'(\alpha))$.
- For each n we have sets $X_0, \ldots, X_n$ so that $M(X_{i+1}, X_i, l)$.
- $\mathsf{T}^{l+1} \vdash \mathsf{Wo}_{\lhd}^{\Pi^0_1(X_j)}(\alpha) \to \mathsf{Wo}_{\lhd}^{X_j}(f(\alpha))$ and $\mathsf{T}^{l+1} \vdash \Pi^0_1(X_j) \dot{\subseteq} X_{j+1}$.

□

From T^l to T^{l+1} – the limit of a theory

Lemma

If f describes T^l from below, then f' describes T^{l+1} from below.

Proof.

- to show: $\mathsf{T}^{l+1} \to \mathsf{Wo}_{\lhd}^{\Pi^1_1}(\alpha) \to \mathsf{Wo}_{\lhd}(f'(\alpha))$.
- For each n we have sets $X_0, \ldots, X_n$ so that $M(X_{i+1}, X_i, l)$.
- $\mathsf{T}^{l+1} \vdash \mathsf{Wo}_{\lhd}^{\Pi^0_1(X_j)}(\alpha) \to \mathsf{Wo}_{\lhd}^{X_j}(f(\alpha)$ and $\mathsf{T}^{l+1} \vdash \Pi^0_1(X_j) \dot{\subseteq} X_{j+1}$.
- $\mathsf{T}^{l+1} \vdash \forall n[\mathsf{Wo}_{\lhd}(\alpha) \to \mathsf{Wo}_{\lhd}(f^n(\alpha))]$.
- $\mathsf{T}^{l+1} \vdash \mathsf{Prog}_{\lhd}(\{\alpha : \mathsf{Wo}_{\lhd}(f'(\alpha))\})$.
- Since $\{\alpha : \mathsf{Wo}_{\lhd}(f'(\alpha))\}$ is Π^1_1, the claim follows.

□

Transfinite reduction

Lemma

Assume that f describes $\overset{*}{\mathsf{T}}^{l}$, that $d \in \mathcal{F}_k$ and that Γ is a finite set of elementary formulas of $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-1})$. Then for all $m \in \mathbb{N}$,

$$\bigcup_{n \in \mathbb{N}} \overset{*}{\mathsf{T}}^{\epsilon, l^n, d} \mathrel{\Big|\!\frac{\alpha}{*}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^{l^m, d} \mathrel{\Big|\!\frac{f'(\alpha)}{*}} \Gamma,$$

where l^m is the composite label $(l, \ldots, l) \in \mathcal{F}_m$.

Transfinite reduction

Lemma

Assume that f describes $\overset{*}{\mathsf{T}}^l$, that $d \in \mathcal{F}_k$ and that Γ is a finite set of elementary formulas of $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-1})$. Then for all $m \in \mathbb{N}$,

$$\bigcup_{n\in\mathbb{N}} \overset{*}{\mathsf{T}}^{\epsilon,l^n,d} \overset{\alpha}{\underset{*}{\vdash}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^{l^m,d} \overset{f'(\alpha)}{\underset{*}{\vdash}} \Gamma,$$

where l^m is the composite label $(l, \ldots, l) \in \mathcal{F}_m$.

Proof: By induction on α

- Last inference is a cut: $\bigcup_{n\in\mathbb{N}} \overset{*}{\mathsf{T}}^{\epsilon,l^n,d} \overset{\alpha'}{\underset{*}{\vdash}} \Gamma, [\neg]A$, where $\alpha' < \alpha$ and $A \in \overset{*}{\mathsf{T}}^{\epsilon,l^{m+k},d}(*)$.
- By I.H. and a cut: $\overset{*}{\mathsf{T}}^{\epsilon,l^{m+k},d} \overset{f'(\alpha')+1}{\underset{*}{\vdash}} \Gamma$.
- f describes $\overset{*}{\mathsf{T}}^l$: $\overset{*}{\mathsf{T}}^{l^m,d} \overset{f^k(f'(\alpha')+1)}{\underset{*}{\vdash}} \Gamma$, this gives $\overset{*}{\mathsf{T}}^{l^m,d} \overset{f'(\alpha'+1)}{\underset{*}{\vdash}} \Gamma$.

From T^{l} to T^{l+1} – the limit of a theory

Lemma

If f describes T^{l} from above, then f' describes T^{l+1} from above.

Proof.

$$\overset{*}{\mathsf{T}}^{l+1,d} \overset{\alpha}{\underset{*}{\vdash}} \Gamma \Longrightarrow \bigcup_{n\in\mathbb{N}} \overset{*}{\mathsf{T}}^{\epsilon,l^n,d} \overset{\alpha}{\underset{*}{\vdash}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^{\epsilon,d} \overset{f'(\alpha)}{\underset{*}{\vdash}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^{d} \overset{f'(\alpha)}{\underset{*}{\vdash}} \Gamma. \qquad \square$$

Π^1_2-Refl$_\mathsf{T}$– From T^{Λ^-} to T^{Λ}

Lemma

BASE $\vdash \breve{\mathsf{T}}^{\Lambda_{\text{head}}} \wedge \breve{\mathsf{T}}^{\Lambda_{\text{tail}}} \rightarrow \breve{\mathsf{T}}^{\Lambda}$. *e.g.* BASE $\vdash \breve{\mathsf{T}}^{1,0,0} \wedge \breve{\mathsf{T}}^{\Omega,0} \rightarrow \breve{\mathsf{T}}^{1,\Omega,0}$.

Proof: By induction on the position of Ω in Λ

- Assume that $\Pi^1_2(Z, e)$.
- Observe that $\breve{\mathsf{T}}^{\Lambda_{\text{head}}}$ is Π^1_2.
- Use $\breve{\mathsf{T}}^{\Lambda_{\text{tail}}}$ to reflect $\Pi^1_2(Z, e) \wedge \breve{\mathsf{T}}^{\Lambda_{\text{head}}}$.
- This yields an X so that $X \models \Pi^1_2(Z, e) \wedge X \models (\breve{\mathsf{T}}^{\Lambda_{\text{head}}} \wedge \breve{\mathsf{T}}^{\Lambda^-_{\text{tail}}})$.
- By I.H. $X \models \breve{\mathsf{T}}^{\Lambda_{\text{head}} \oplus \Lambda^-_{\text{tail}}}$, i.e. $X \models \breve{\mathsf{T}}^{\Lambda^-}$.

Reducing Π^1_2-reflection from below

Lemma

$\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}^{\Pi^1_2}_{\lhd}(\alpha) \rightarrow \breve{\mathsf{T}}^{\Lambda[\alpha]}$.

Proof that $\mathsf{Prog}_{\lhd}\{\alpha : \breve{\mathsf{T}}^{\Lambda[\alpha]}\}$.

- Recall that $\breve{\mathsf{T}}^{\Lambda[\alpha]}$ is $\forall Z \exists X M(X, Z, \Lambda[\alpha])$ with $M(U, V, l)$ a Σ^1_1 formula.
- Recall: $\Lambda[\alpha{+}1] = (\Lambda[\alpha] \oplus \Lambda^{-}_{\mathsf{tail}}){+}1$.
- By the previous lemma: $\mathsf{T}^{\Lambda} \vdash \breve{\mathsf{T}}^{\Lambda[\alpha] \oplus \Lambda^{-}_{\mathsf{tail}}}$.
- $\mathsf{T}^{\Lambda} \vdash \breve{\mathsf{T}}^{l} \rightarrow \breve{\mathsf{T}}^{l+1}$ (since T^{Λ} entails $(\Sigma^1_1$-DC$)$).

Reducing Π^1_2-reflection from above

Lemma

For $c \in \mathcal{F}_{k-1}$ *and each finite set* Γ *of* $\mathsf{L}^*(\mathsf{M}_0, \ldots, \mathsf{M}_{k-2})$ *formulas, we have*

$$\overset{*}{\mathsf{T}}^{\Lambda,c} \overset{\alpha}{\underset{*}{\vdash}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^{\Lambda[\alpha],c} \overset{\alpha}{\underset{*}{\vdash}} \Gamma.$$

Proving the Main Theorem by transfinite induction on $(<, \mathsf{I}^{\Omega})$

Theorem (Main Theorem)

For each k-label $0 < l$ that is not a hard limit, and each hard limit Λ, we have

- $\alpha \mapsto \varphi^{k+1} l\alpha$ *describes* T^{l}.
- $\alpha \mapsto \varphi^{k+1} \Lambda[\alpha]0$ *describes* T^{Λ}.

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Proof.

- $\overset{*}{\mathsf{T}}{}^{\Lambda,c} \vdash_{*}^{<\alpha} \Gamma$, then $\overset{*}{\mathsf{T}}{}^{\Lambda[\alpha'],c} \vdash_{*}^{<\alpha} \Gamma$. Now by I.H. $\overset{*}{\mathsf{T}}{}^{\epsilon,c} \vdash_{*}^{<\varphi^{k+1}\Lambda[\alpha]0} \Gamma$.

□

Proving the Main Theorem by transfinite induction on $(<, \mathsf{I}^{\Omega})$

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For each k-label $0 < l$ that is not a hard limit, and each hard limit Λ, we have

- $\alpha \mapsto \varphi^{k+1} l \alpha$ *describes* T^{l}.
- $\alpha \mapsto \varphi^{k+1} \Lambda[\alpha] 0$ *describes* T^{Λ}.

Proof.

- $\overset{*}{\mathsf{T}}^{\Lambda,c} \mathrel{\big|\!\frac{<\alpha}{*}} \Gamma$, then $\overset{*}{\mathsf{T}}^{\Lambda[\alpha'],c} \mathrel{\big|\!\frac{<\alpha}{*}} \Gamma$. Now by I.H. $\overset{*}{\mathsf{T}}^{\epsilon,c} \mathrel{\big|\!\frac{<\varphi^{k+1}\Lambda[\alpha]0}{*}} \Gamma$.
- $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}^{\Pi^1_2}_{\lhd}(\alpha) \rightarrow \breve{\mathsf{T}}^{\Lambda[\alpha]}$ and $\mathsf{T}^{\Lambda[\alpha]} \vdash \mathsf{Wo}^{\Pi^1_2}_{\lhd}(\beta) \rightarrow \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[\alpha]\beta)$.
- Hence $\mathsf{T}^{\Lambda} \vdash \mathsf{Wo}^{\Pi^1_2}_{\lhd}(\alpha) \rightarrow \mathsf{Wo}_{\lhd}(\varphi^{k+1}\Lambda[\alpha]0)$.

□

An application: An upper bound for ID^*_α

ID^*_1

1. For each $L_1(\mathsf{P})$ formula $A(\mathsf{P}^+, u)$, the constant Fix^A is a fixed point of the operator $F^A(X) \mapsto \{x : A(X, x)\}$ associated with $A(\mathsf{P}^+, u)$.
2. For each $L_1(\mathsf{P})$ formula $B(\mathsf{P}^+, u)$, if $\mathcal{C} := \{x : B(\mathsf{Fix}^A, x)\}$, then

$$F^A(\mathcal{C}) \subseteq \mathcal{C} \rightarrow \mathsf{Fix}^A \subseteq \mathcal{C}.$$

Lemma

For each arithmetical $A(\mathsf{P}^+, u)$, Σ^1_1-DC_0 proves:

$$\mathsf{Fix}^A := \bigcap \{X : \mathsf{F}^A(X) \subseteq X\} \text{ is the least } \Pi^1_1\text{-definable fixed point of } \mathsf{F}^A.$$

An application: An upper bound for ID^*_α, cont.

- $\mathsf{ID}^*_{\lhd\gamma}$ is straightforwardly embedded into a theory T with

$$\text{if}\, \gamma' \lhd \gamma, \text{ then } \mathsf{T} \vdash \exists F\, \mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, \gamma').$$

- Consider the case $\gamma = \omega^\alpha + \omega^\beta$.

An application: An upper bound for ID^*_α, cont.

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- Consider the case $\gamma = \omega^\alpha + \omega^\beta$.
- Recall that if $\gamma' \lhd \omega^\alpha + \omega^\beta$, then

$$\mathsf{T}^{1,\beta}[\mathsf{T}^{1,\alpha}] \vdash \exists F\, \mathsf{CHSeq}^{\mathsf{T}^\Omega}(F, \omega^\alpha + \omega^\beta).$$

- $|\mathsf{ID}^*_{\lhd\omega^\alpha+\omega^\beta}| \leq \varphi 1\alpha(\varphi 1\beta 0)$.

The formula $M(X,Z,l)$

- We set $M(X,Z,l) := \exists Y I(X,Y,Z,l)$.
- $I(X,Y,Z,l)$ express that X is a transitive model of T^l above Z.
- For each $l' \prec l$, $(Y)_{l'}$ contains pairs $\langle e,f\rangle$.
- $\langle e,f\rangle \in (Y)_{l'}$ means "$(X)_e$ is a model of $\mathsf{T}^{l'}$ above $(X)_f$", e.g.

$$
\begin{aligned}
\langle e,f\rangle \in (Y)_{l'} \quad & \text{iff} \quad \forall f' \exists e' [(X)_{f'} \dot{\in} (X)_e \to (X)_{e'} \dot{\in} (X_e) \wedge \langle e',f'\rangle \in (Y)_{l'}] \\
\langle e,f\rangle \in (Y)_{\Lambda} \quad & \text{iff} \quad \forall f', x, \exists e' [(X)_{f'} \dot{\in} (X)_e \wedge \Pi^1_2((X)_{f'}, x) \restriction (X)_e \to \\
& \qquad (X)_{e'} \dot{\in} (X)_e \wedge \langle e',f'\rangle \in (Y)_{\Lambda^-} \wedge \Pi^1_2((X)_{f'}, x) \restriction (X)_{e'})].
\end{aligned}
$$

- Globel closure conditions.

Outline

Summary

We have a modular ordinal analysis for systems below Π^1_3 reflection on ω-models of ACA_0. For theories generated by the operations lim (l) and Π^1_2-Refl_T (p), we have

$$\begin{aligned}
\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_0} &:= l^{\alpha_0} \circ (l \circ p^1)^{\alpha_1} \circ \ldots \circ (l \circ p^{k-1})^{\alpha_{k-1}}\ (\Pi^0_1\text{-}\mathsf{CA}_0^-) \\
|\mathsf{T}^{\alpha_{k-1},\ldots,\alpha_0}| &= \varphi^{k+1}\alpha_{k-1}\ldots\alpha_0 0. \\
\mathsf{T}^{\alpha_{k-1},\ldots,\Omega,\ldots,0} &:= p^{i+1} \circ (l \circ p^{i+1})^{\alpha_{i+1}} \circ \ldots \circ (l \circ p^{k-1})^{\alpha_{k-1}}\ (\Pi^0_1\text{-}\mathsf{CA}_0^-) \\
|\mathsf{T}^{\alpha_{k-1},\ldots,\Omega,\ldots,0}| &= \varphi^{k+1}\alpha_{k-1}\ldots\alpha_{i+1}\omega 0\ldots 0
\end{aligned}$$

Example

- T^Ω is equivalent to Σ^1_1-DC_0. $|\mathsf{T}^\Omega| = \varphi\omega 0$.
- $\mathsf{T}^{1,0}$ is $\lim(\Sigma^1_1\text{-}\mathsf{DC}_0)$ or ATR_0. $|\mathsf{T}^{1,0}| = \varphi 100 = \Gamma_0$.
- $\mathsf{T}^{\Omega,0}$ is Π^1_2-$\mathsf{Refl}_{\Sigma^1_1\text{-}\mathsf{DC}}$ and $|\mathsf{T}^{\Omega,0}| = \varphi\omega 00$.

Future work

- It remains to generalize this modular ordinal analysis to Π^1_n reflection on ω-models of ACA_0. This yields a ordinal analysis of ID_1 by metapredicative methods.

Future work

- It remains to generalize this modular ordinal analysis to Π^1_n reflection on ω-models of ACA_0. This yields a ordinal analysis of ID_1 by metapredicative methods.
- Provide an interface to subsystems of Kripke-Platek set theory and explicit mathematics.

Roadmap: Ordinal analysis of theories up to ID_1 by metapredicative methods