Extended Predicative Universes

Part II: Π_3-Reflection

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(Joint work with R. Kahle)

1. Problems of the direct Π_3-reflecting Universe.
2. The constructed Π_3-reflecting universe.
3. The extended predicative Π_3-reflecting universe.
4. Conclusion.

Work in progress

- Work in progress.
- A.S. specialist in MLTT (Martin-Löf Type Theory) but not in explicit mathematics.
- We model an extended predicative version of the Π_3-reflecting universe according to the Π_3-reflecting universe in type theory.
- No model or proof theoretic analysis yet done (only done for the version in MLTT).

1. Direct Π_3-Reflecting Universe

- Notations
  - Let $\Re$ be the collection of names (analogue of Set in MLTT).
  - We write $x \in a$ instead of $x \dot{\in} a$.
  - $\Re_\Re := \{x \in \Re \mid \forall y \in x.y \in \Re\}$.

Simple, Super, Mahlo Universes

- A universe is an $\mathrm{v} \in \Re_{\Re}$ closed upwards and downwards under universe operations (strictness).
- A super universe (Palmgren) is
  - a universe v
  - s.t. for $a \in \mathrm{v}$ there exists a universe $\mathrm{su}\ a$ s.t.
    - $a \in \mathrm{su}\ a$,
    - $\mathrm{su}\ a \in \mathrm{v}$.
- A Mahlo universe (S.) is
  - a universe v
  - s.t. for $a \in \mathrm{v}$ and $f \in \mathrm{v} \to \mathrm{v}$ there exists a universe $\mathrm{su}\ f\ a$ s.t.
    - $a \in \mathrm{su}\ f\ a$,
    - $f \in \mathrm{su}\ f\ a \to \mathrm{su}\ f\ a$,
    - $\mathrm{su}\ f\ a \in \mathrm{v}$.

Direct Π_3-Refl. Universe

- Idea for Π_3-reflecting universe:
- A Π_3-reflecting universe (Jäger, Strahm) is ‘
  - a universe v
  - s.t. for
    - $a \in \mathrm{v}$,
    - $f \in \mathrm{v} \to \mathrm{v}$,
    - $F \in (\mathrm{v} \to \mathrm{v}) \to (\mathrm{v} \to \mathrm{v})$

    there exists $\mathrm{su}\ F\ f\ a$ s.t.
    - $a \in \mathrm{su}\ F\ f\ a$,
    - $f \in \mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a$,
    - $F \in (\mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a)$
      $\to (\mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a)$
    - $\mathrm{su}\ F\ f\ a \in \mathrm{v}$.
- **Advantage:** Very short and concise.

Problem of Direct Π_3-Refl. Univ

- We have

$$F \in (\mathrm{v} \to \mathrm{v}) \to (\mathrm{v} \to \mathrm{v})$$

and demand

$$F \in (\mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a) \\ \to (\mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a)$$

- But a $g \in \mathrm{su}\ F\ f\ a \to \mathrm{su}\ F\ f\ a$ is not an element of $\mathrm{v} \to \mathrm{v}$ so the type of F doesn't provide a reason for allowing F to be applied to g.
- The existence of a Π_3-reflecting universe demands that $\mathrm{su}\ F\ f\ a$ is big enough that when computing $F\ g$ we need to refer only to $g\ x$ for $x \in \mathrm{su}\ F\ f\ a$.

Problem of Direct Π_3-Refl. Univ

- If we consider $\mathrm{su}\ F\ f\ a$ as defined by some inductive process we need that if when trying to evaluate $F\ g$ we need to compute $g\ x$, then x needs to be added to $\mathrm{su}\ F\ f\ a$.
- This is difficult to control.
- Therefore it is not suitable for a fully extended predicative analysis.
- A similar problems occurs when carrying out a proof theoretic analysis.
- We follow the development of the ordinal notation system for Π_3-reflection.

2. The constructed Π_3-Reflecting Ur

- First step towards the constructed Π_3-reflecting universe: Hyper-Mahlo.
- A universe v is a **Hyper-Mahlo universe**, if
  - for every $f \in \mathrm{v} \to \mathrm{v}$, $a \in \mathrm{v}$
  - there exists a subuniverse $\mathrm{u}\ 1\ a\ f$ which is
    - Mahlo,
    - closed under a, f,
    - contained in v,
    - and represented in v.
  - $\mathrm{v}' := \mathrm{u}\ 1\ a\ f$ Mahlo means that
    - for every $g \in \mathrm{v}' \to \mathrm{v}'$ and $b \in \mathrm{v}'$
    - there exists a subuniverse $\mathrm{u}\ 0\ b\ g$
    - which is closed under b, g.
    - and represented in v'.

Illustration of the Hyper Mahlo Univ

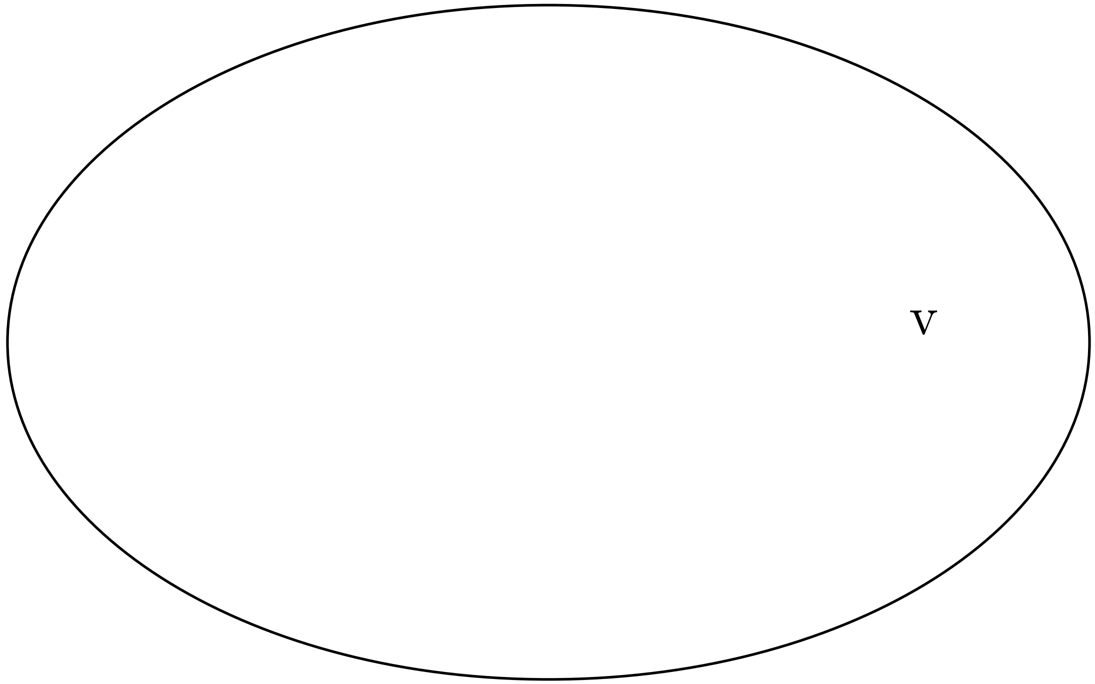
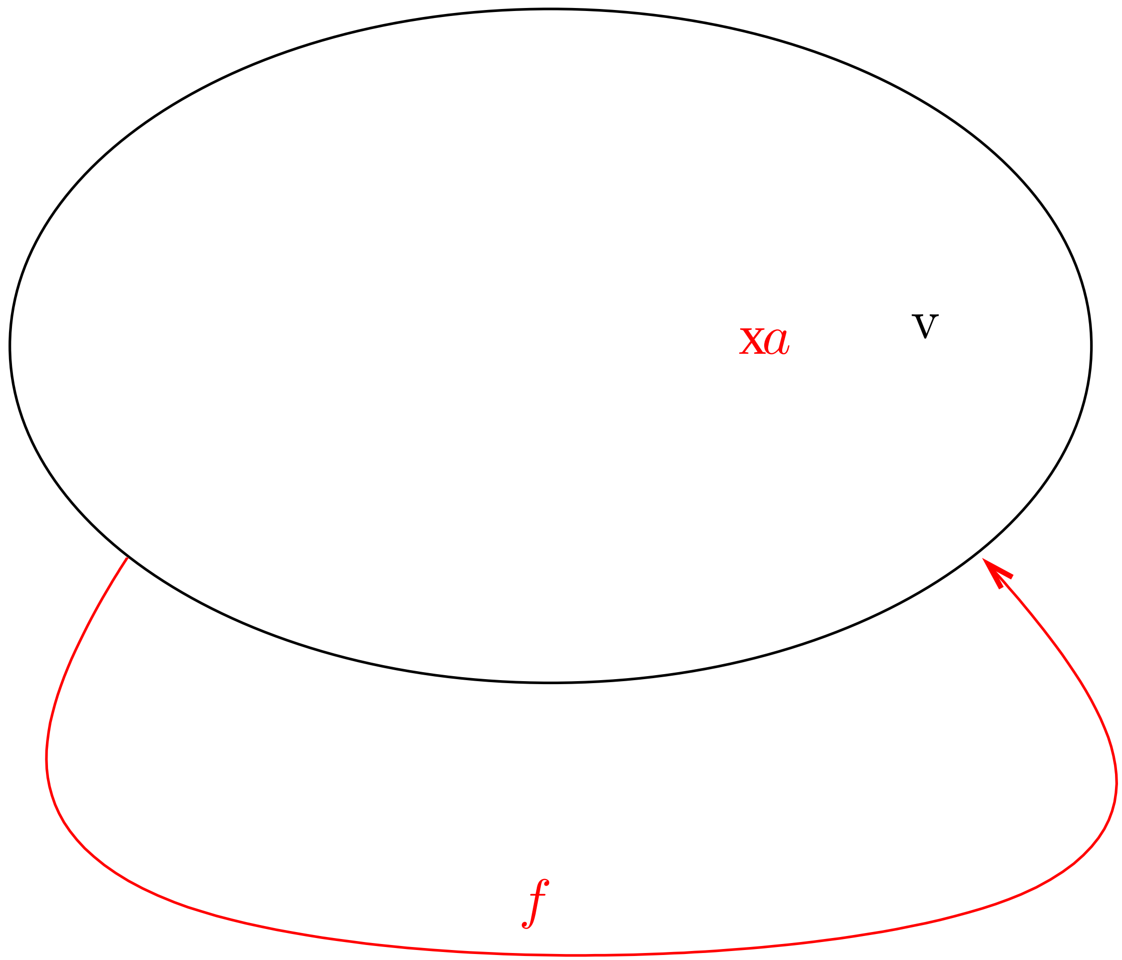


Illustration of the Hyper Mahlo Univ



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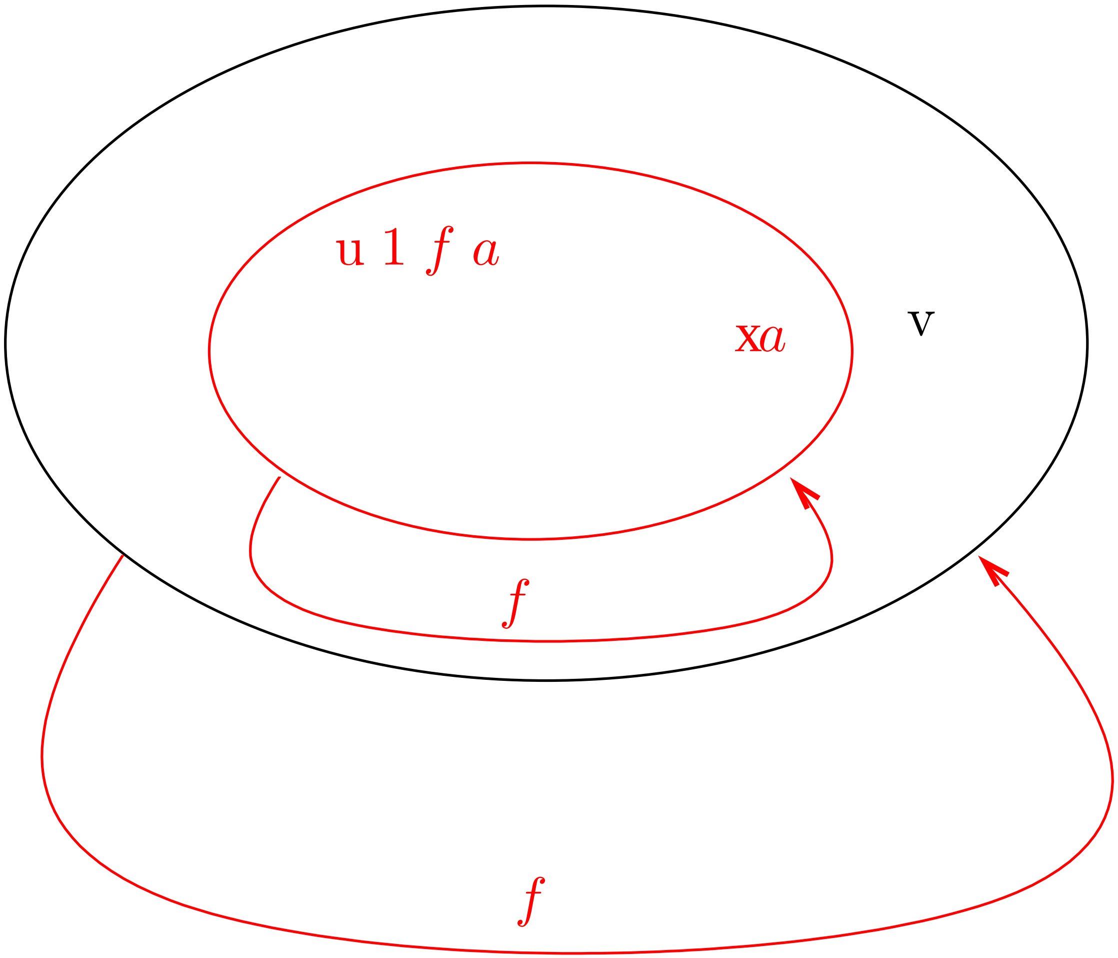

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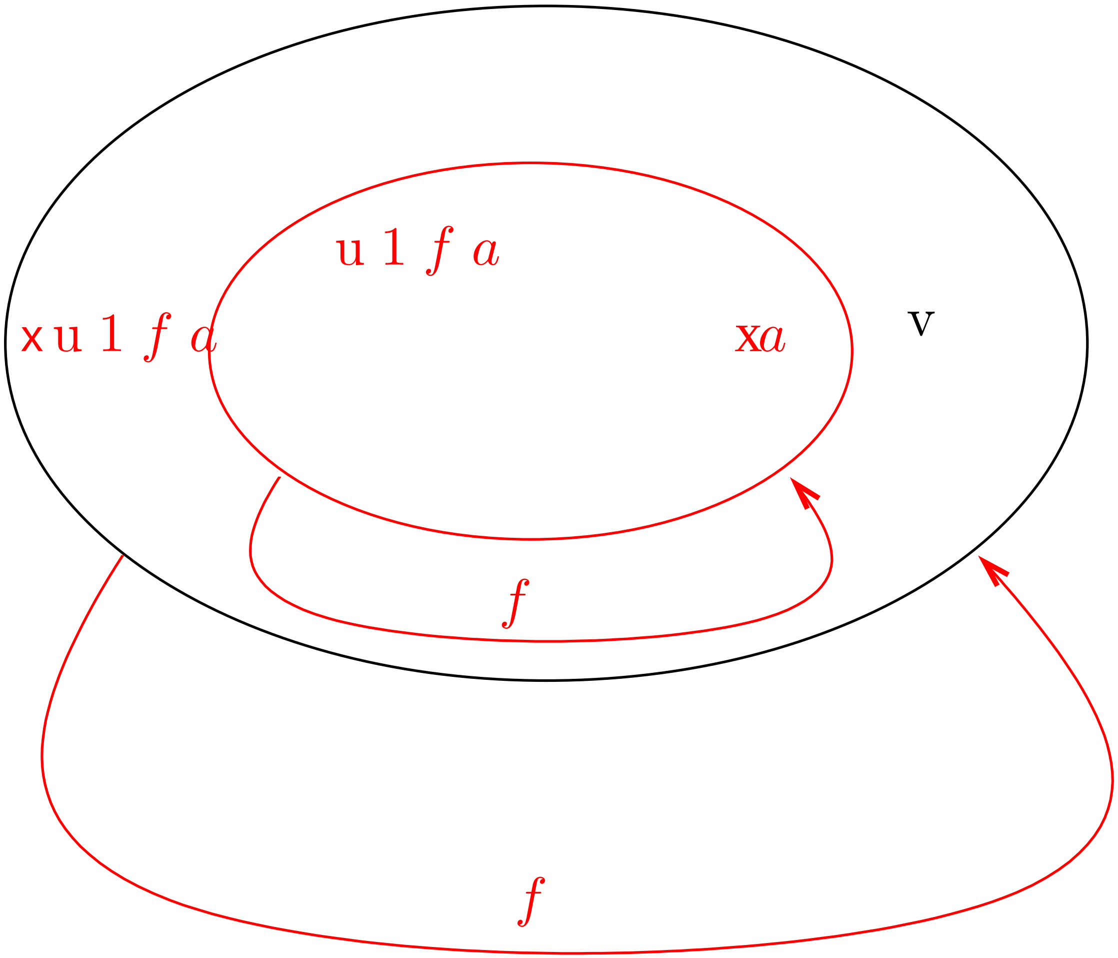

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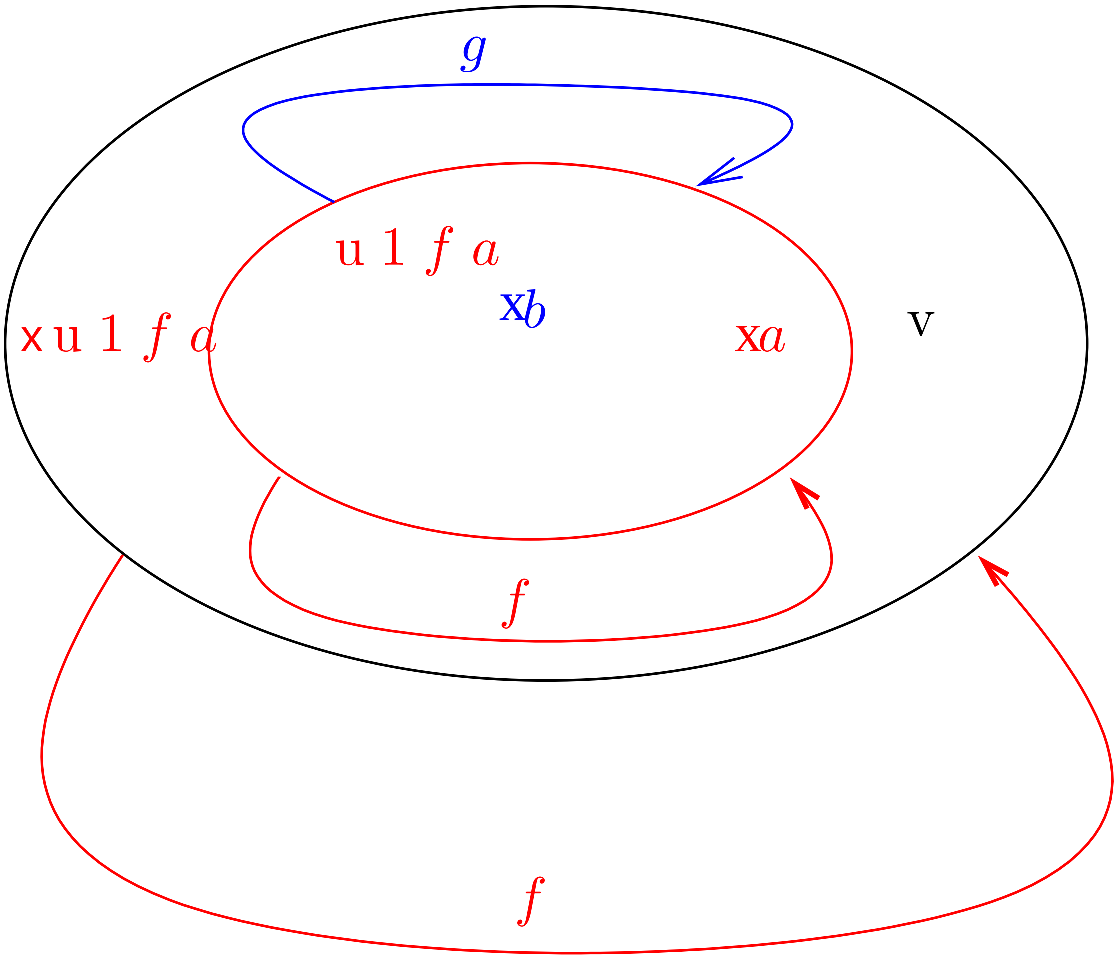

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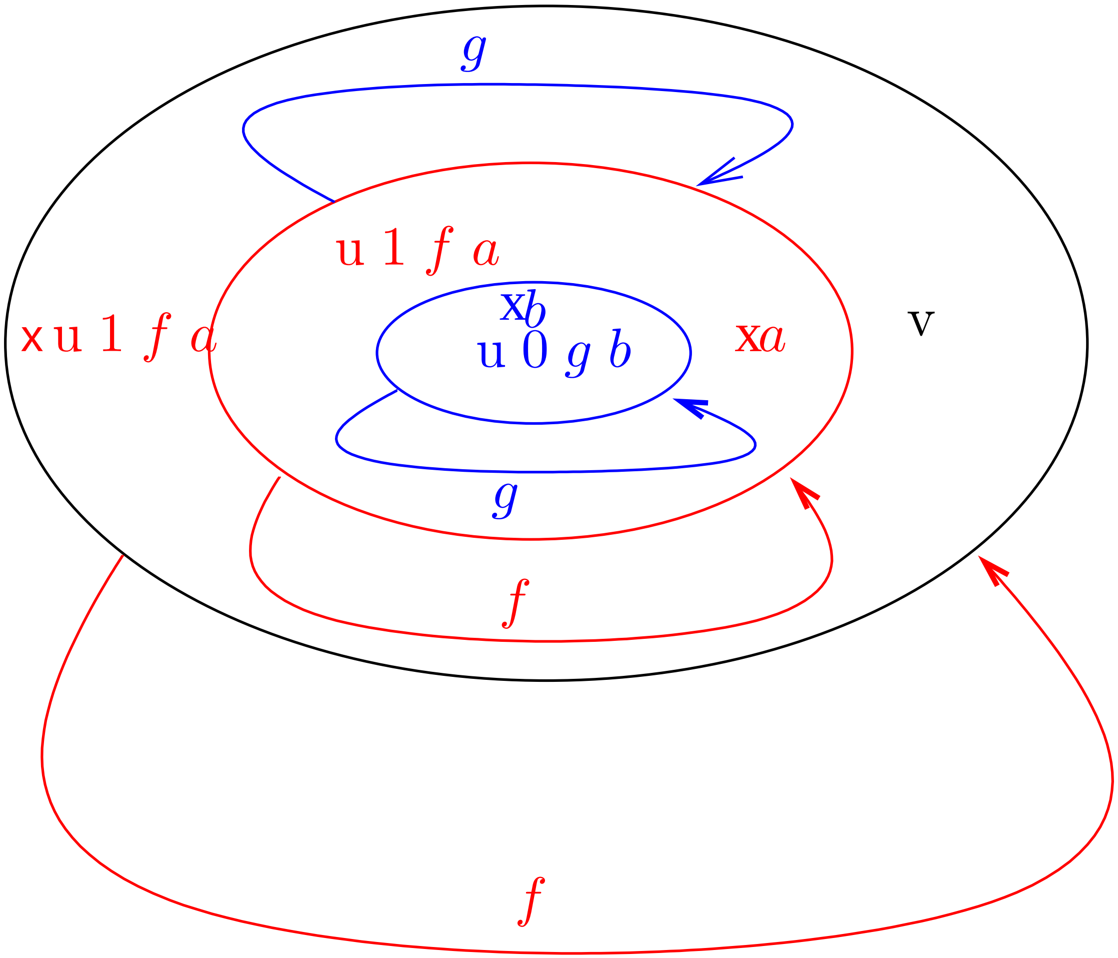

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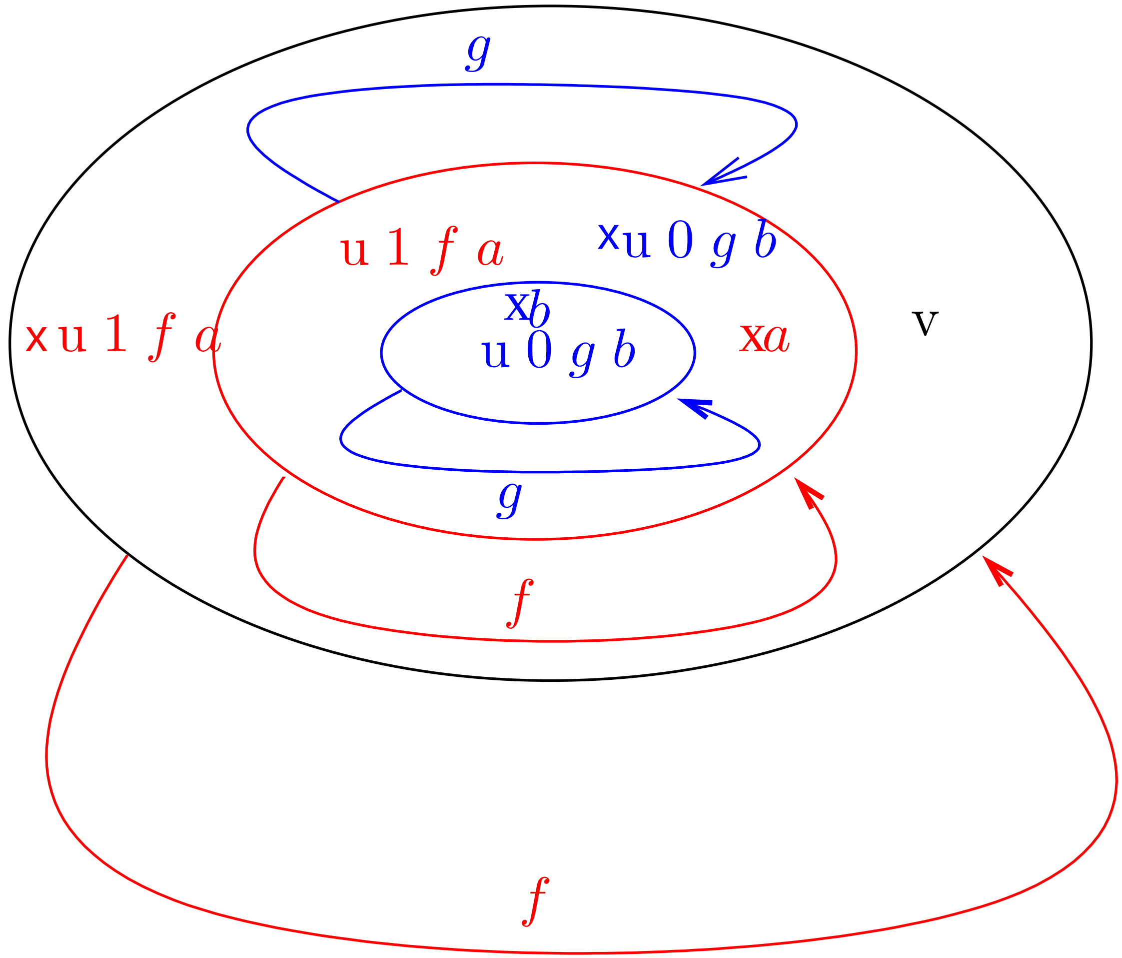


Hyper$^\alpha$ Mahlo Universes

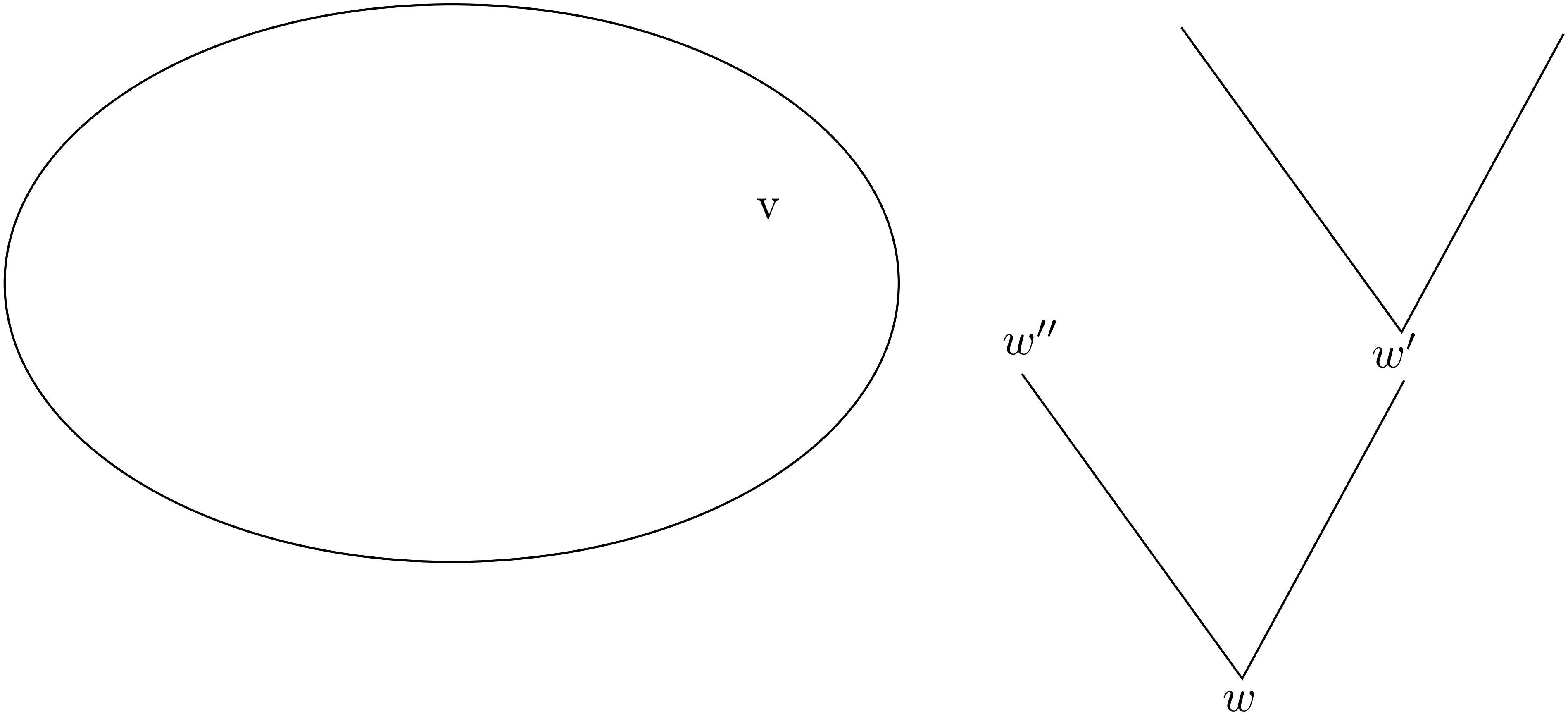
- Can be generalised easily to Hyper$^\alpha$-Mahlo universes.
  - v is **Hyper$^\alpha$-Mahlo**, if for every
    - $\beta < \alpha$,
    - and $a : \mathrm{v}$, $f : \mathrm{v} \to \mathrm{v}$,

    there exists a
    - Hyper$^\beta$-subuniverse of v,
    - closed under a, f,
    - and represented in v.

Autonomously Mahlo Universe (S.)

- Next step is to form a universe v s.t.
  - for every $f \in \mathrm{v} \to \mathrm{v}$, $a \in \mathrm{v}$
  - and for every $w \in \mathrm{w}(\mathrm{v}, \lambda x.x)$ (or its analogue in explicit mathematics)
  - there exists a Hyper^{w}-Mahlo subuniverse of v,
    - i.e. Hyper$^{\alpha}$, where α = height of w.
  - closed under f, a,
  - represented in v.
  - In fact $\mathrm{w}(\mathrm{v}, \lambda x.x)$ is defined simultaneously with v.

Illustration, Autonom. Mahlo Univ.



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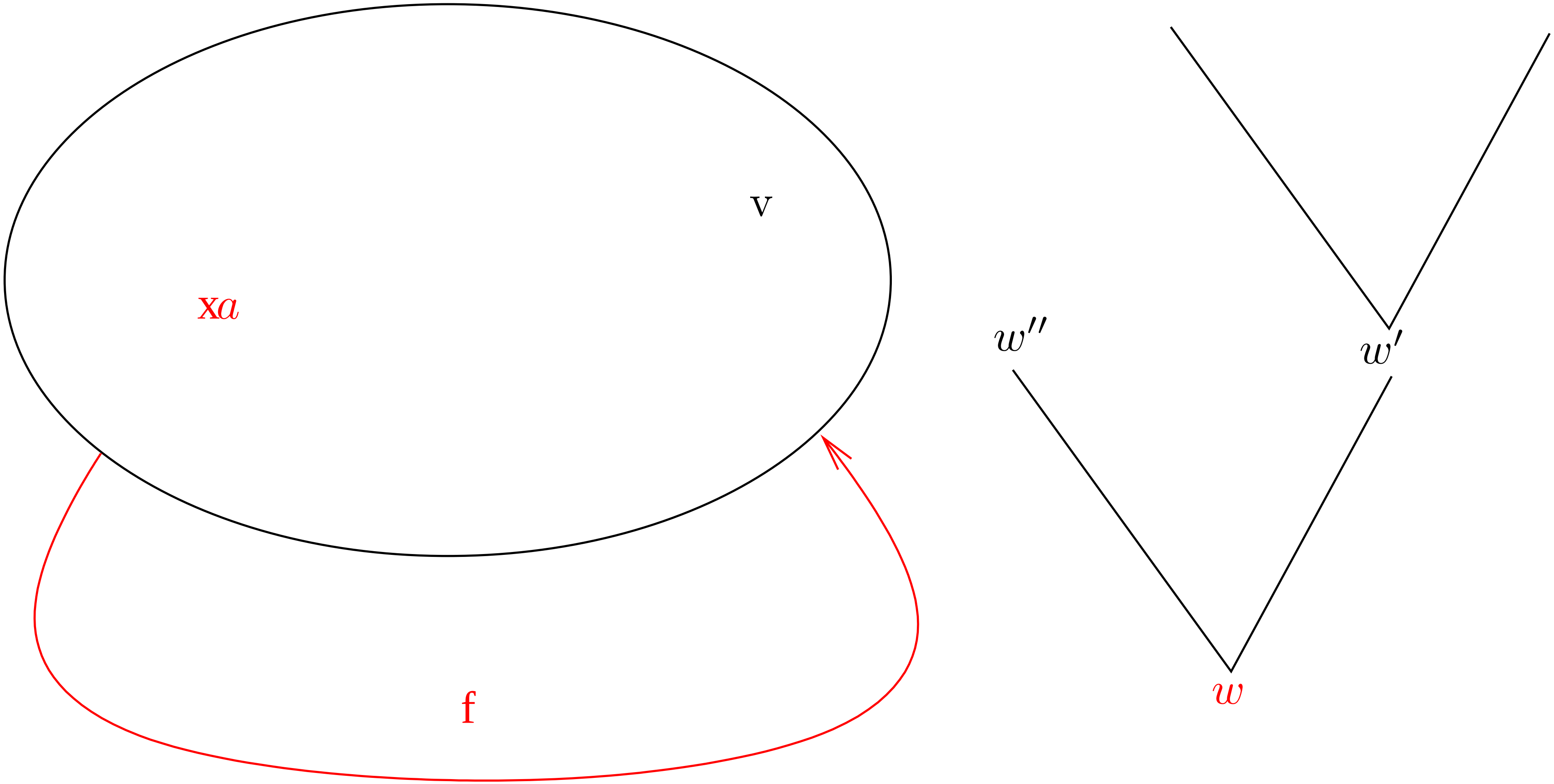

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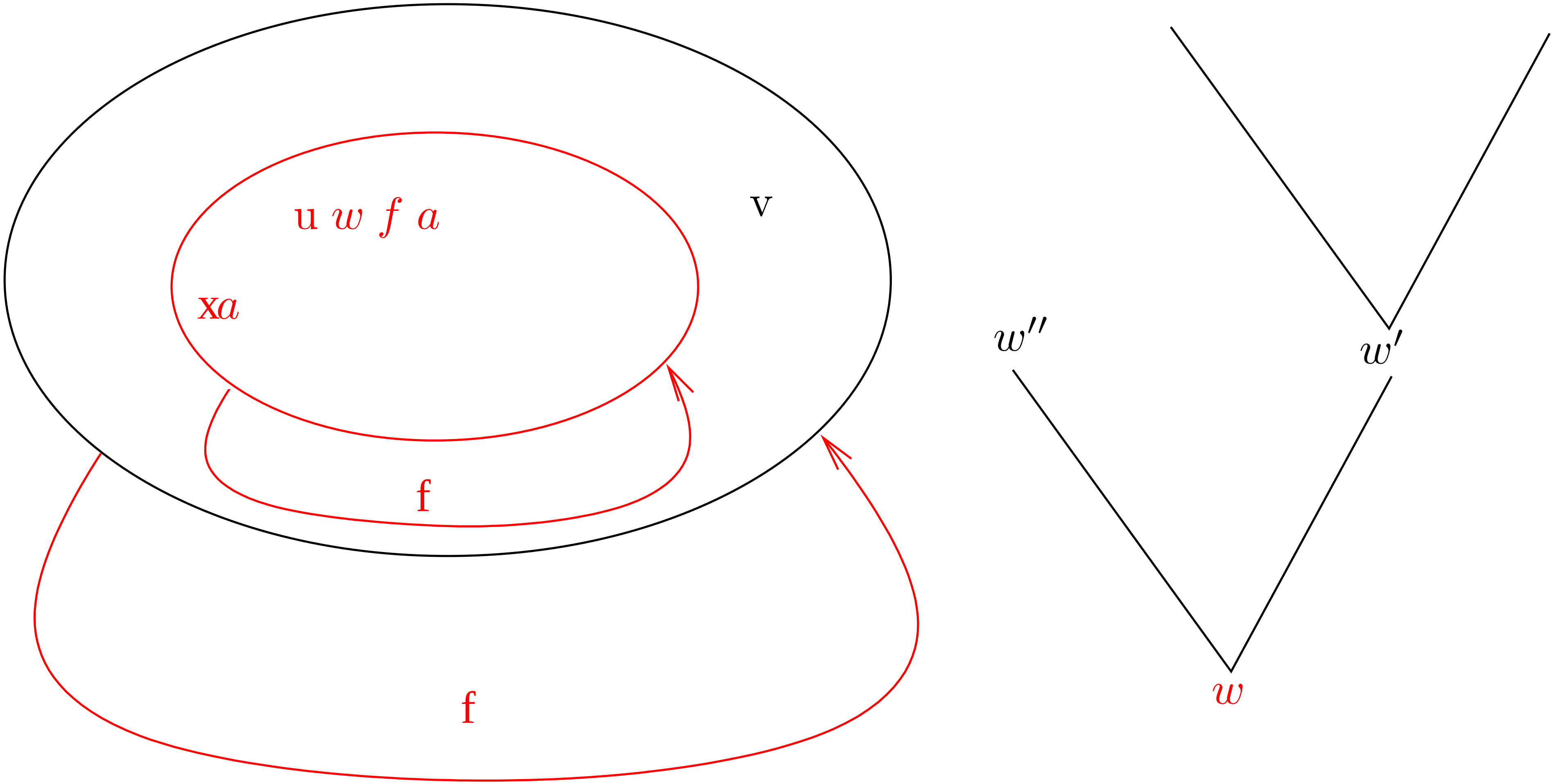

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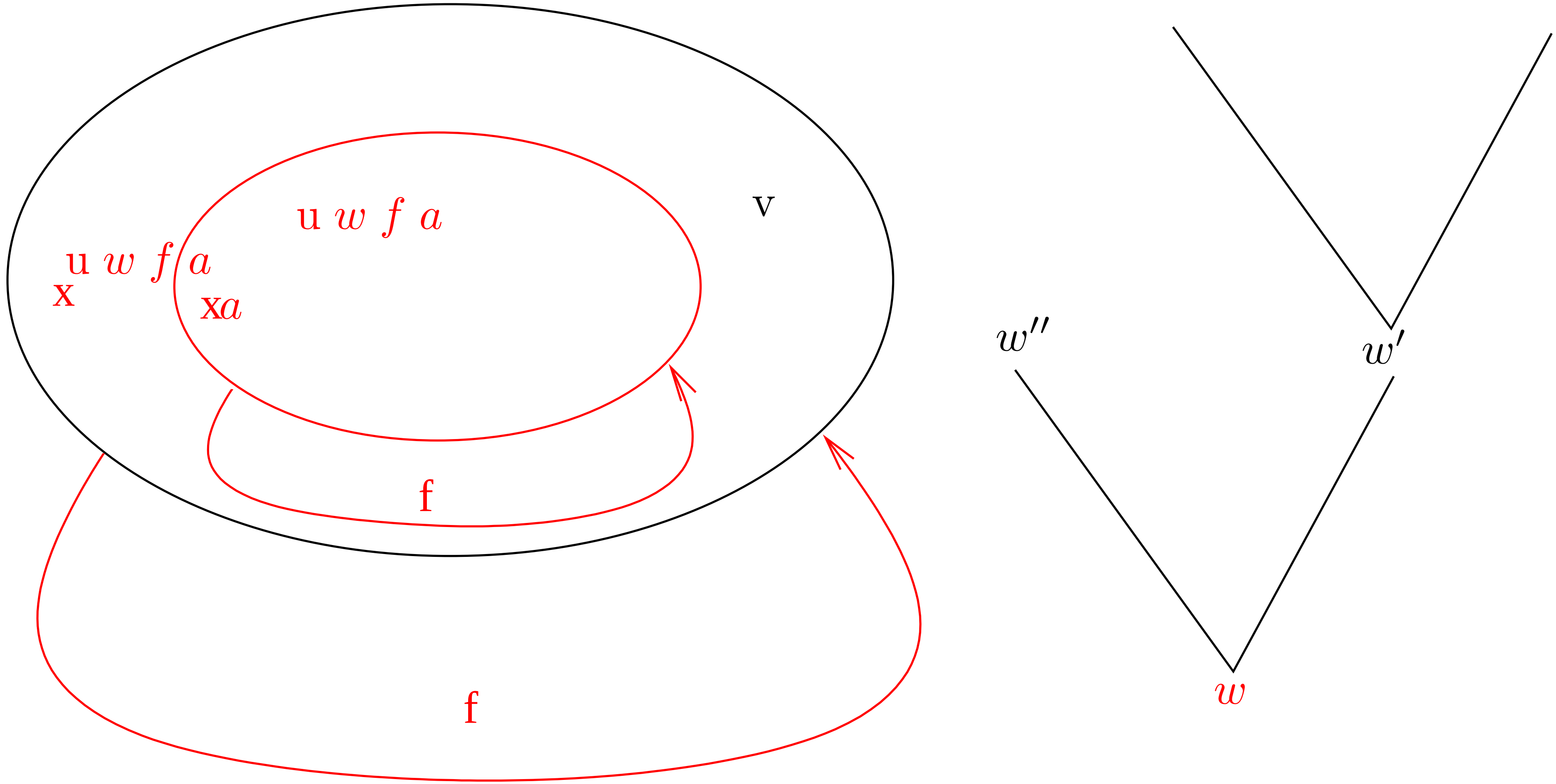

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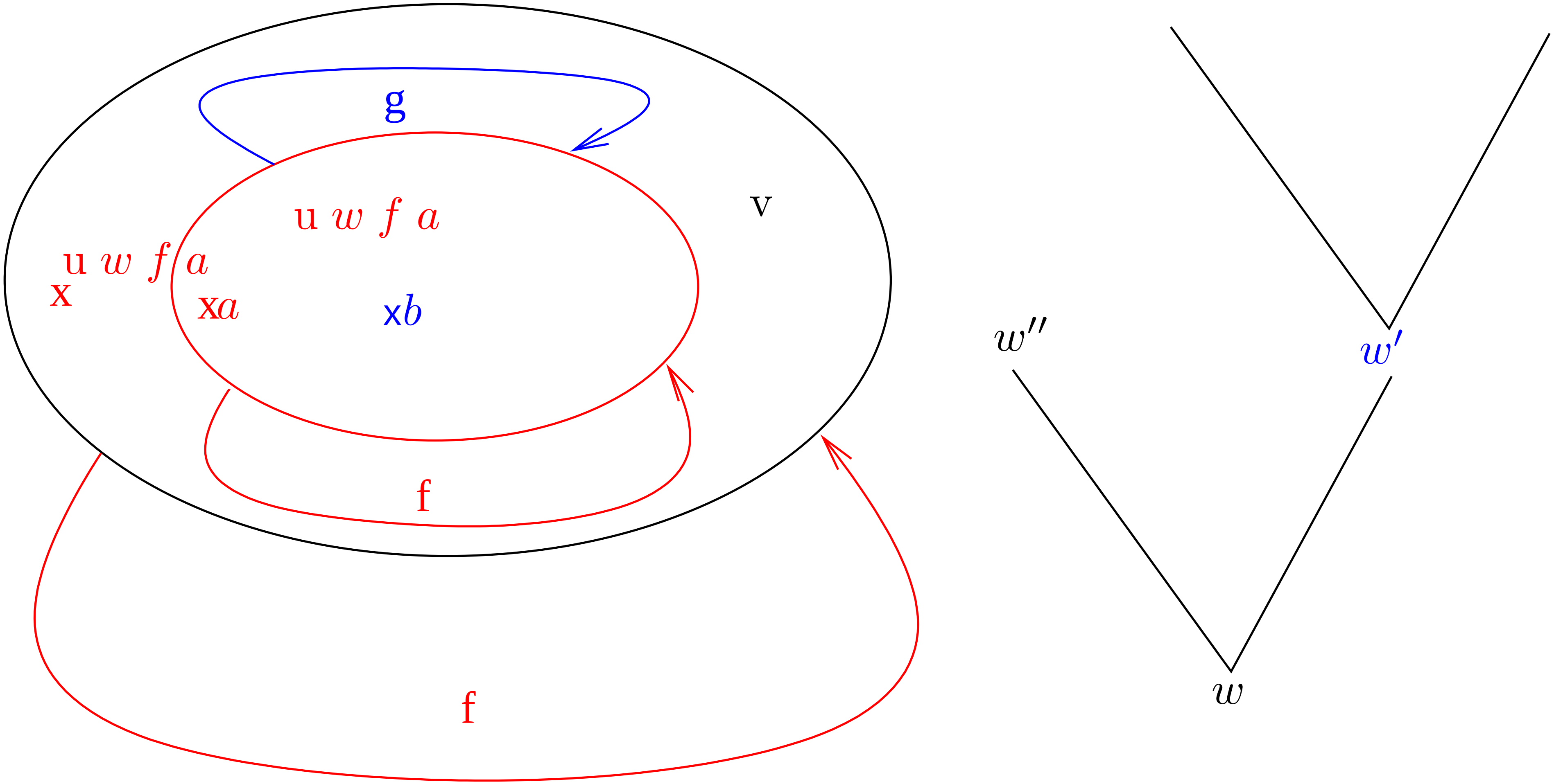

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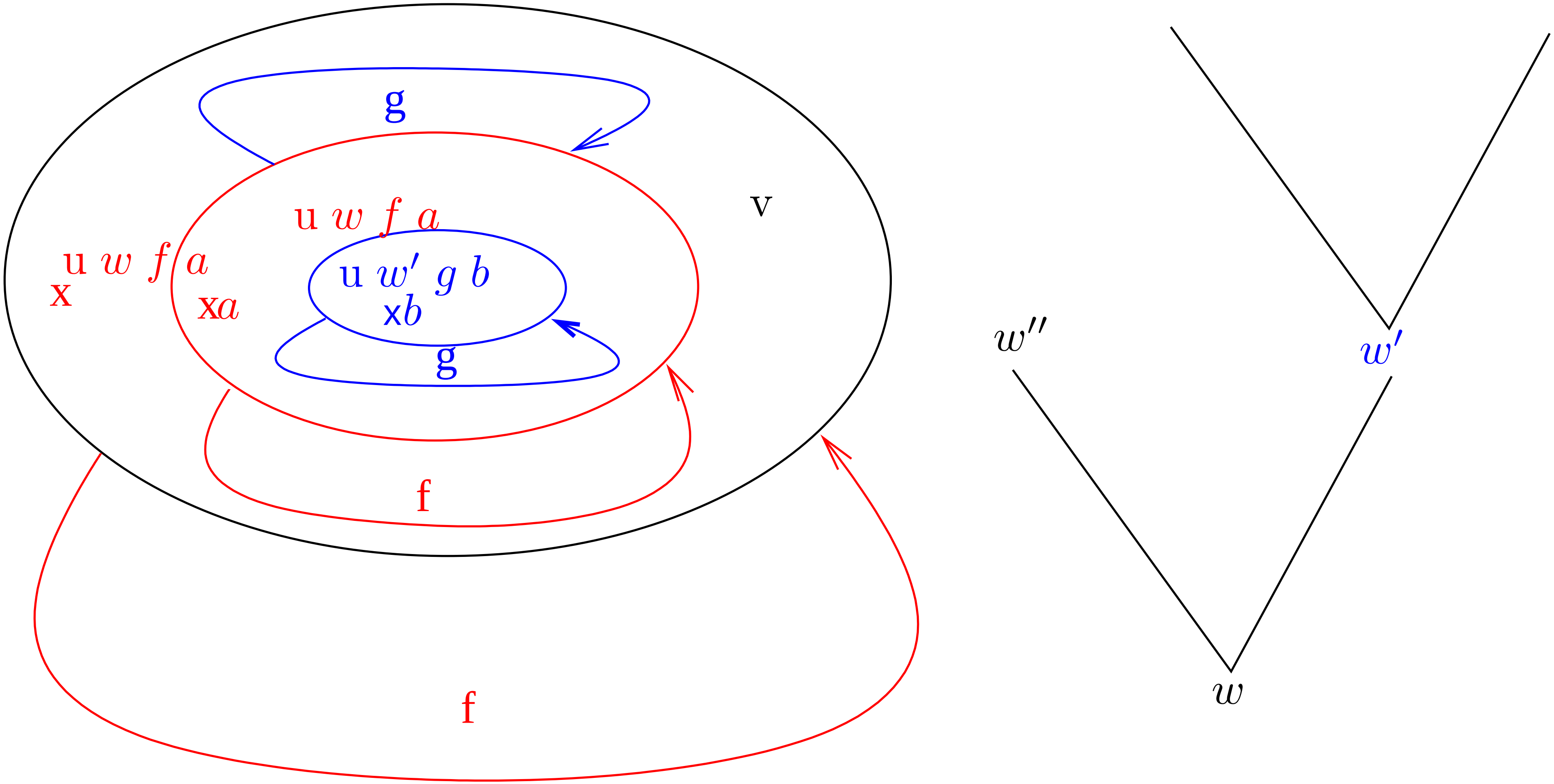

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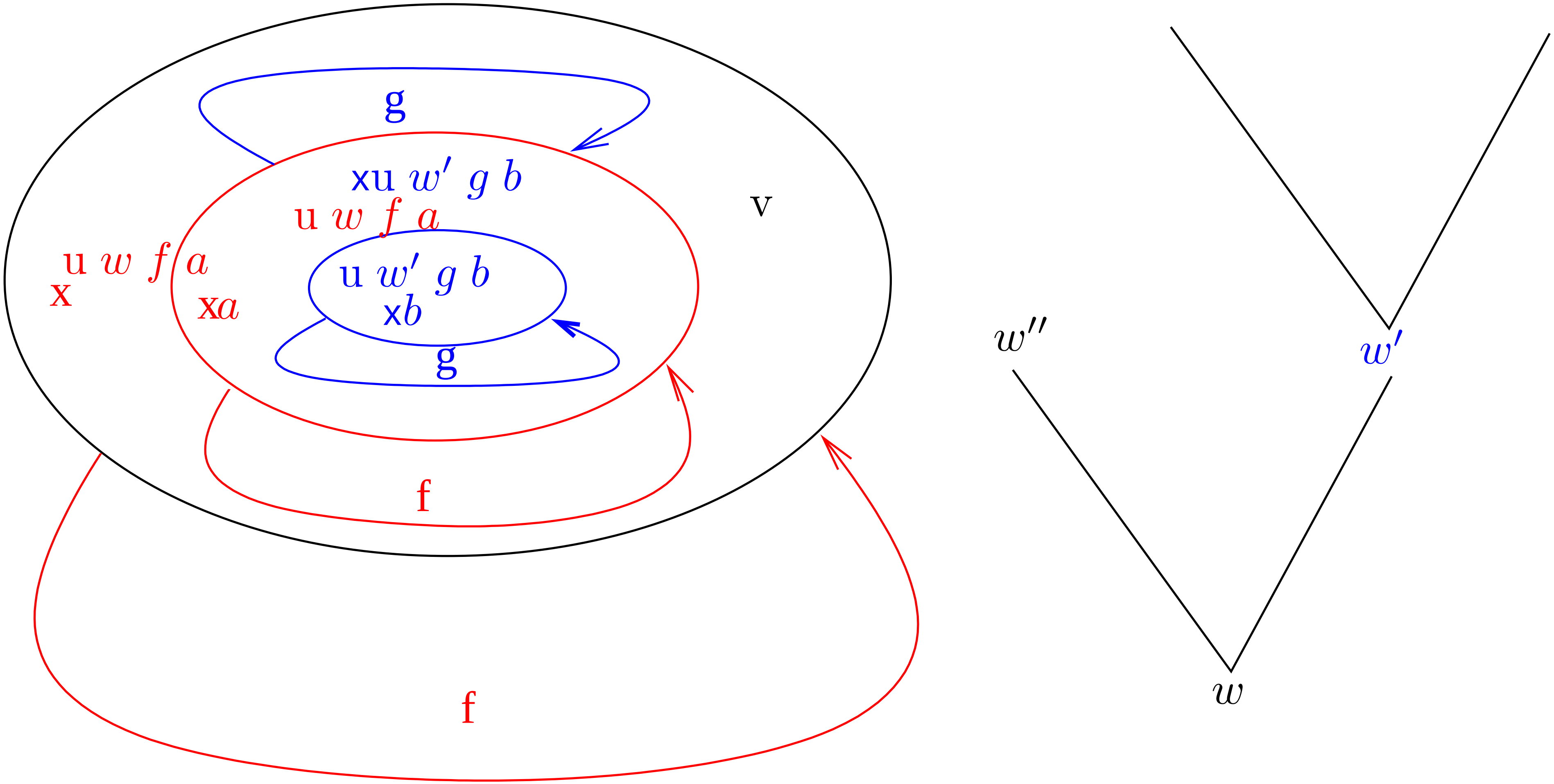


Mahlo Degrees

- With a Hyper$w$-Mahlo universe we can associate the **Mahlo degree** w.
- With the autonomously Mahlo universe itself, we can no longer associate a Mahlo degree, which is an ordinal.
  - The number of subdegrees depends on the autonomously Mahlo universe.
  - But in a **monotone** way and only **locally**:
    - When v increases, $\mathrm{w}(\mathrm{v}, \lambda x.x)$ increases essentially.
    - For every $p \in \mathrm{w}(\mathrm{v}, \lambda x.x)$ we can find $a \in \mathrm{v}$, $b \in a \to \mathrm{v}$ s.t. p is in $\mathrm{w}(a, b)$.

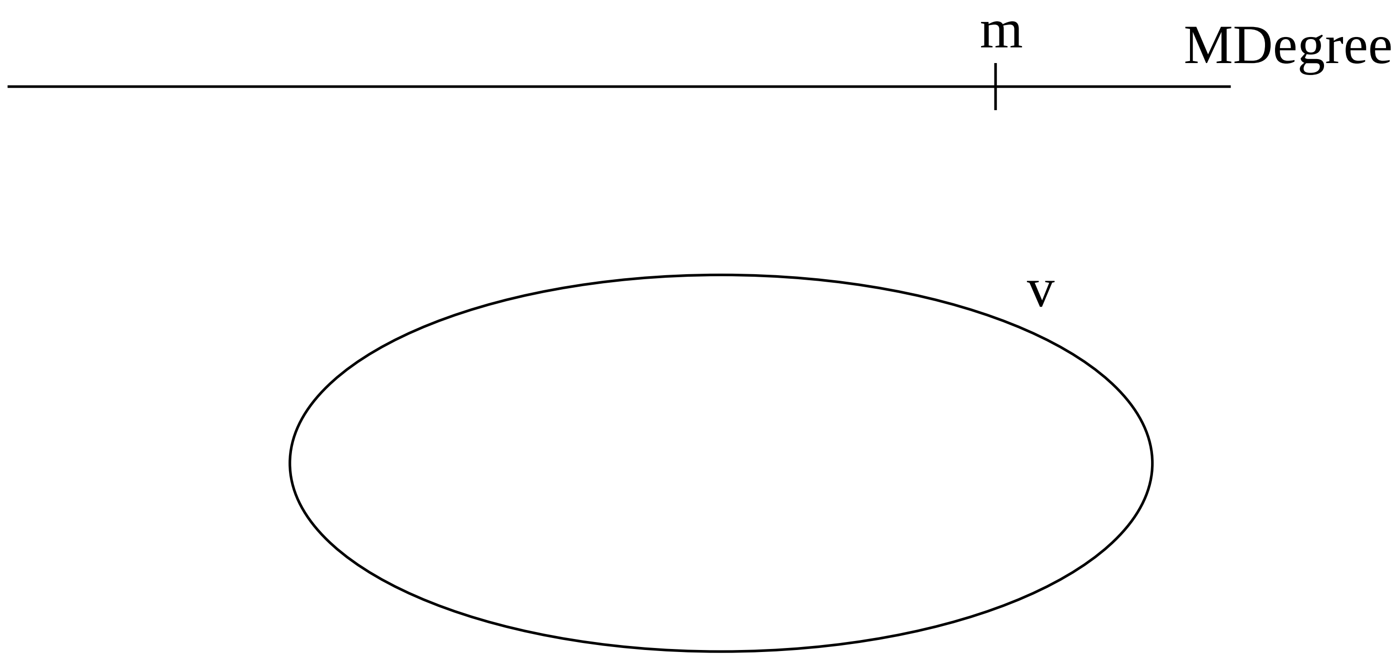
Mahlo Degrees

- So we get: Mahlo degrees on a universe v are given by
  - a set $\mathrm{MDegree}$ of Mahlo degrees,
  - and a function
    $\mathrm{subdeg} : \mathrm{MDegree} \to \mathrm{v} \to \mathrm{Fam}(\mathrm{v}, \mathrm{MDegree})$, which

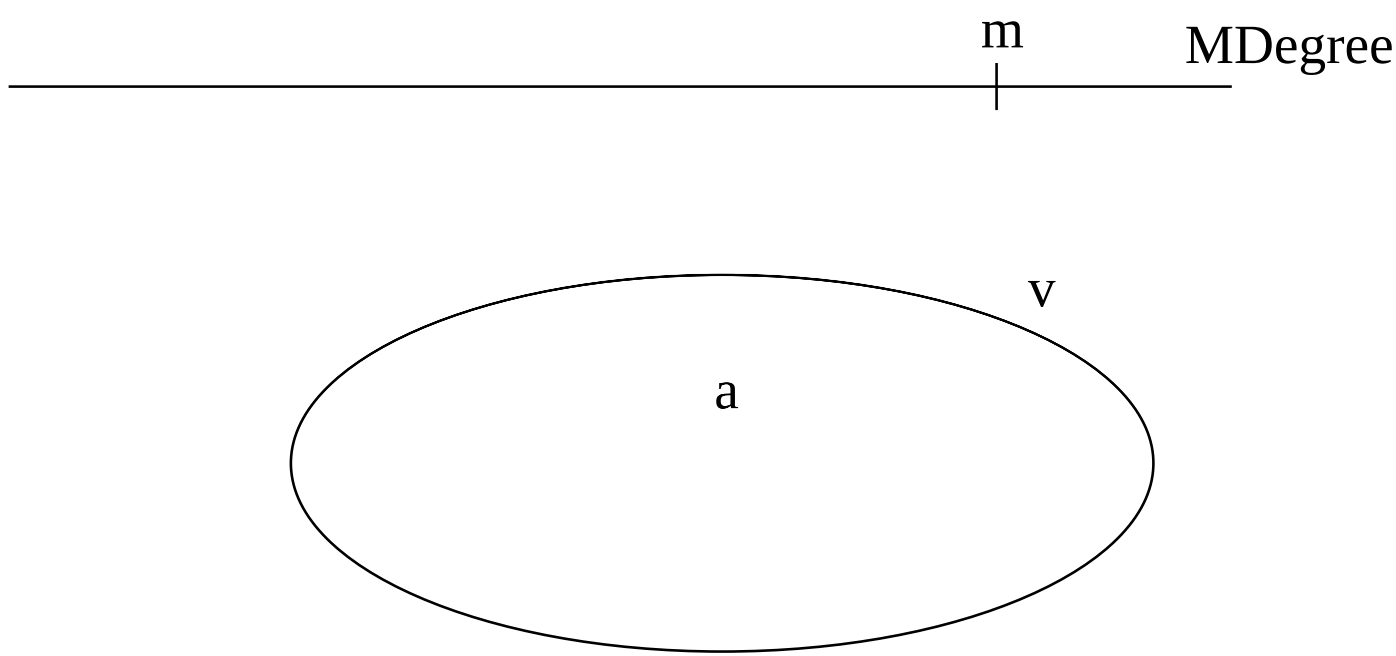
    associates
    - with every Mahlo degree m
    - and $a \in \mathrm{v}$

    a
  - family of Mahlo degrees $\mathrm{subdeg}\, m\, a$ indexed over an element of v.
- In fact we can obtain $d \in \mathrm{MDegree}$ then
  $d \in \mathrm{v} \to \mathrm{Fam}(\mathrm{v}, \mathrm{MDegree})$.

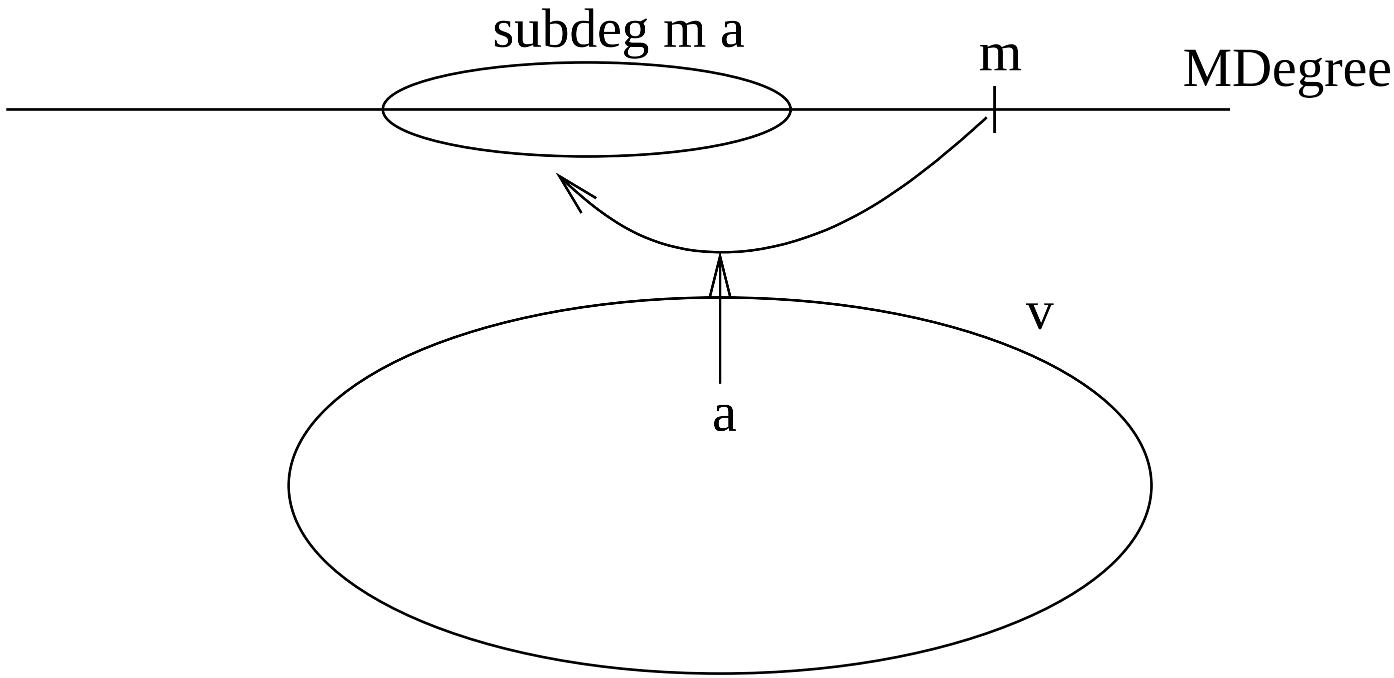
Mahlo Degrees



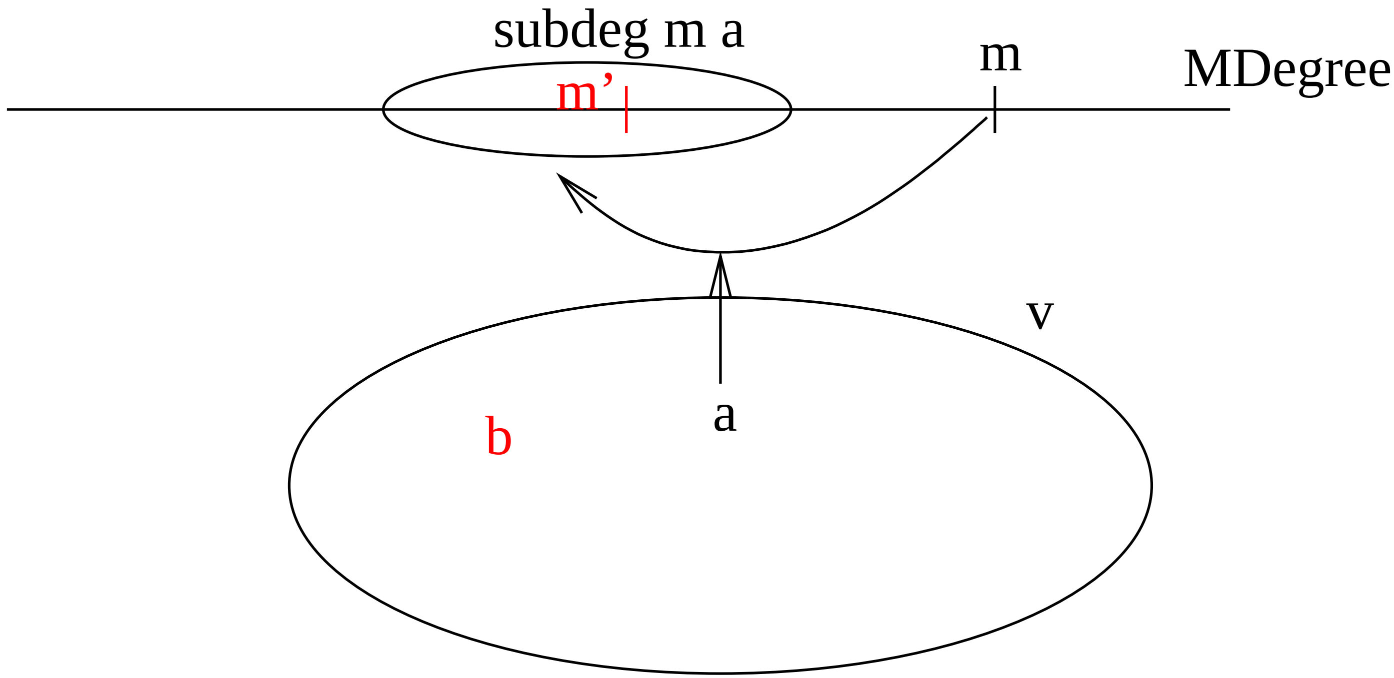
Mahlo Degrees



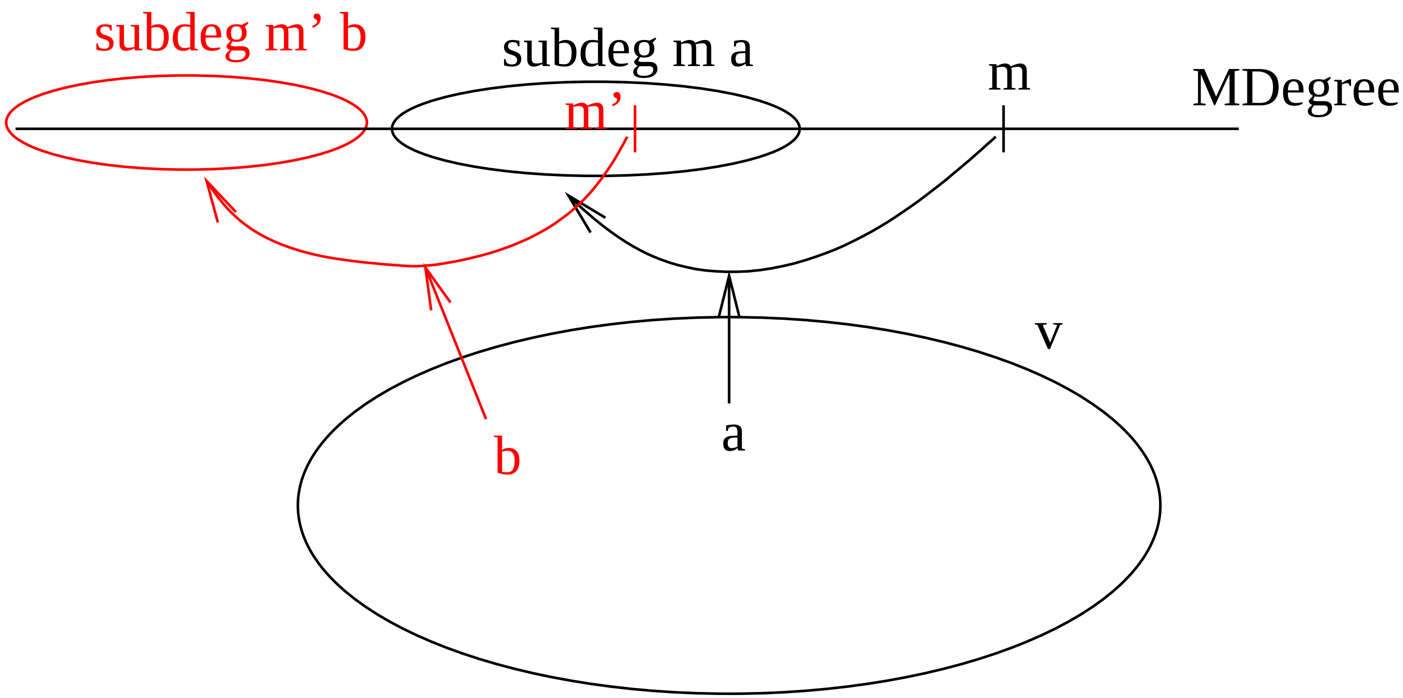
Mahlo Degrees



Mahlo Degrees



Mahlo Degrees



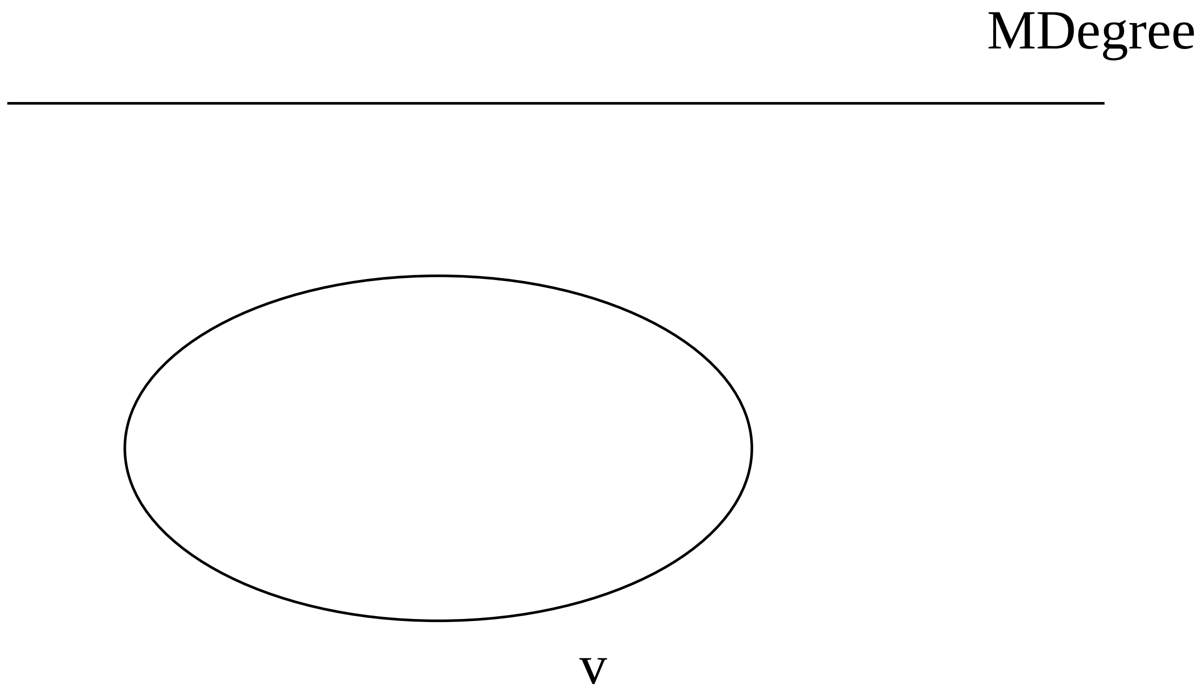
Examples of Mahlo Degrees

- For ordinary Mahlo degrees corresponding to ordinals α we have
  - $\mathrm{subdeg}\,\alpha\,x = \{\beta \mid \beta < \alpha\}$.
- The **autonomously Mahlo universe** has Mahlo degree m, s.t.
  - $\mathrm{subdeg}\,m\,x = \{w \mid w \in \mathrm{w}(y, z)\}$.
    (y, z extracted from x).

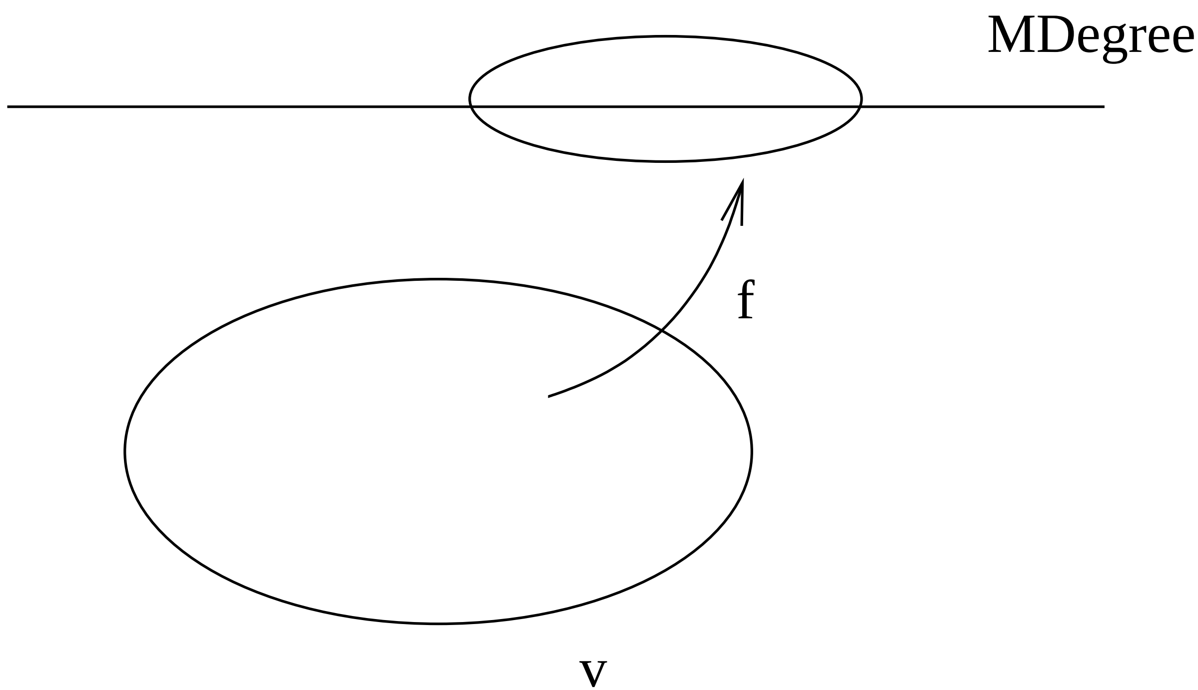
Introduction of Mahlo Degrees

- Introduction rules for Mahlo degrees for the Π_3-reflecting universe v:
  - For every $f : \mathrm{v} \to \mathrm{Fam}(\mathrm{v}, \mathrm{M})$
  - there exists a Mahlo degree $m : \mathrm{MDegree}$
  - s.t. $\mathrm{subdeg}\, m\, x = f\ x$.

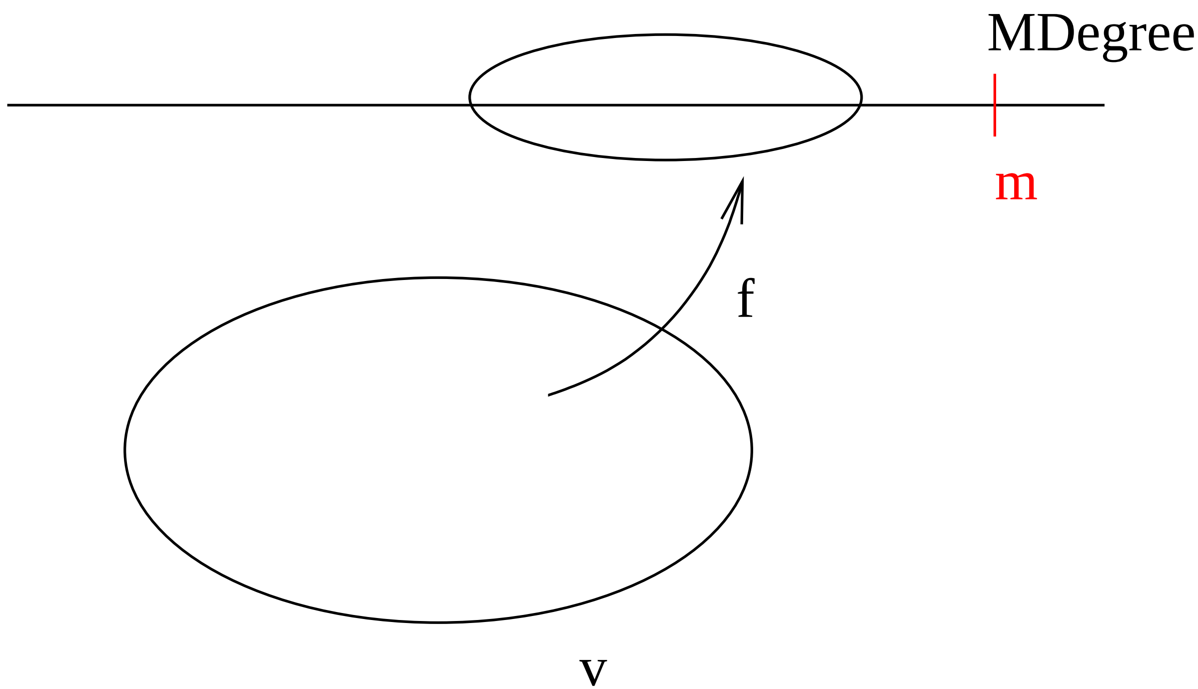
Introduction of Mahlo Degrees



Introduction of Mahlo Degrees



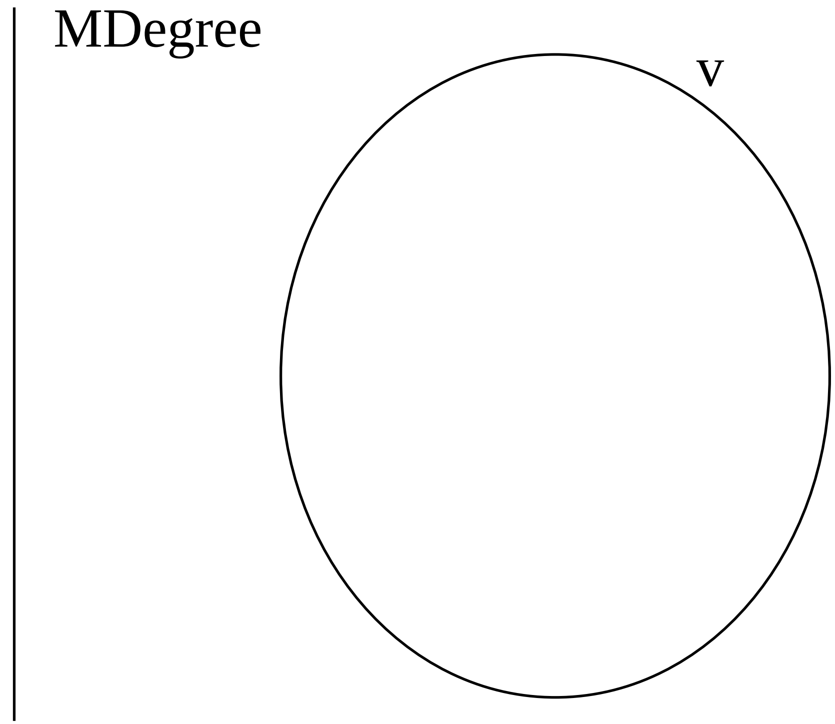
Introduction of Mahlo Degrees



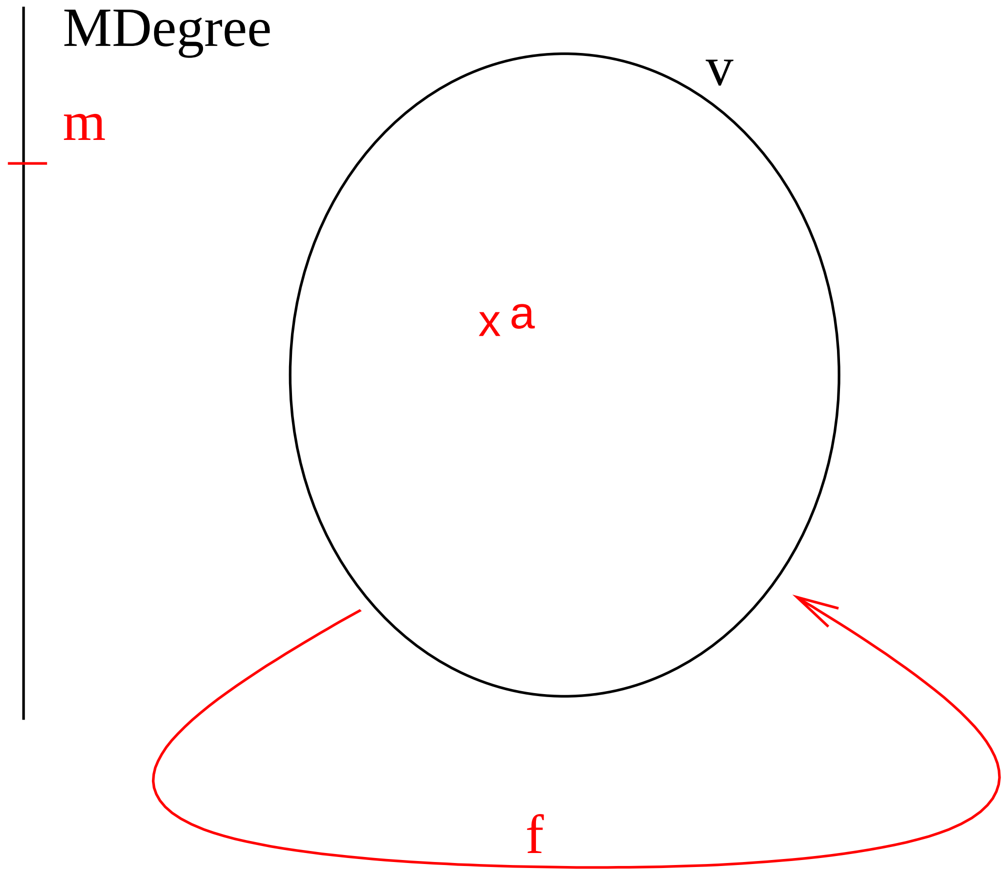
Closure of the Π_3-Refl. Universe (S.)

- Closure of the Π_3-reflecting universe v:
  - For every $m : \mathrm{MDegree}$, $f \in \mathrm{v} \to \mathrm{v}$, $a \in \mathrm{v}$
  - there exists a subuniverse $\mathrm{u}\ f\ m\ a$
  - of Mahlo degree m,
  - closed under f,
  - and containing a,
  - s.t. $\mathrm{u}\ f\ m\ a \in \mathrm{v}$.

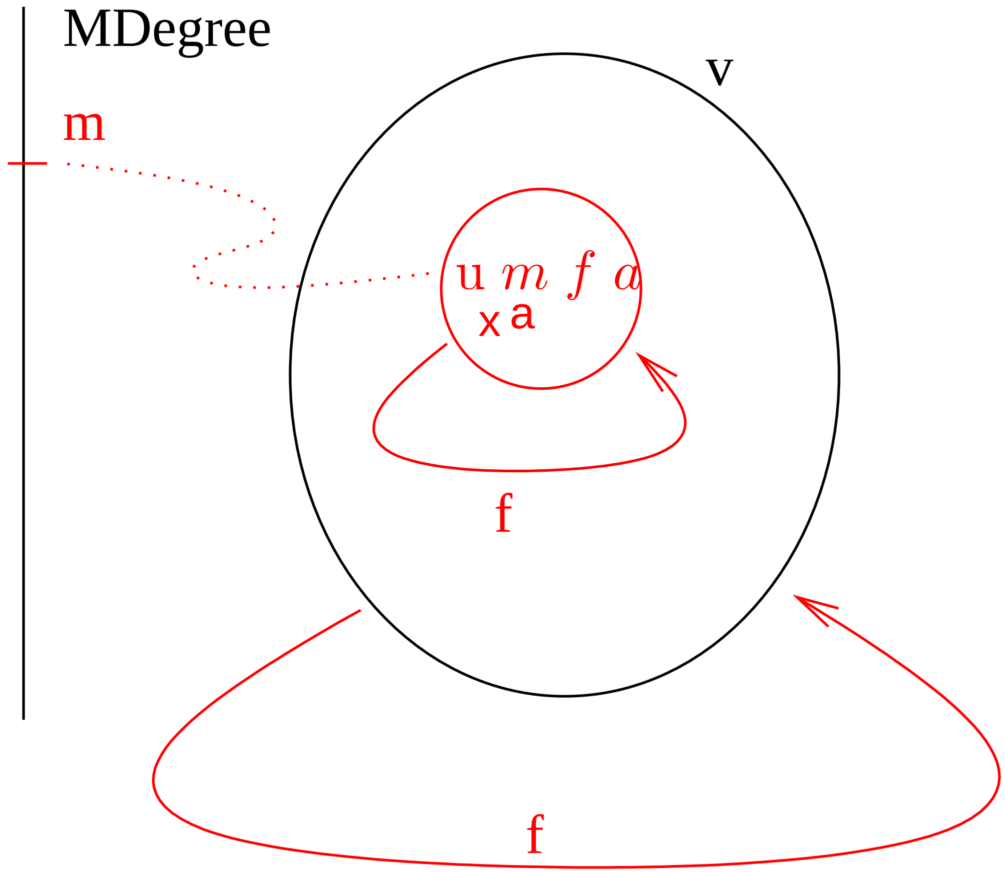
Closure of the Π_3-Refl. Univ.



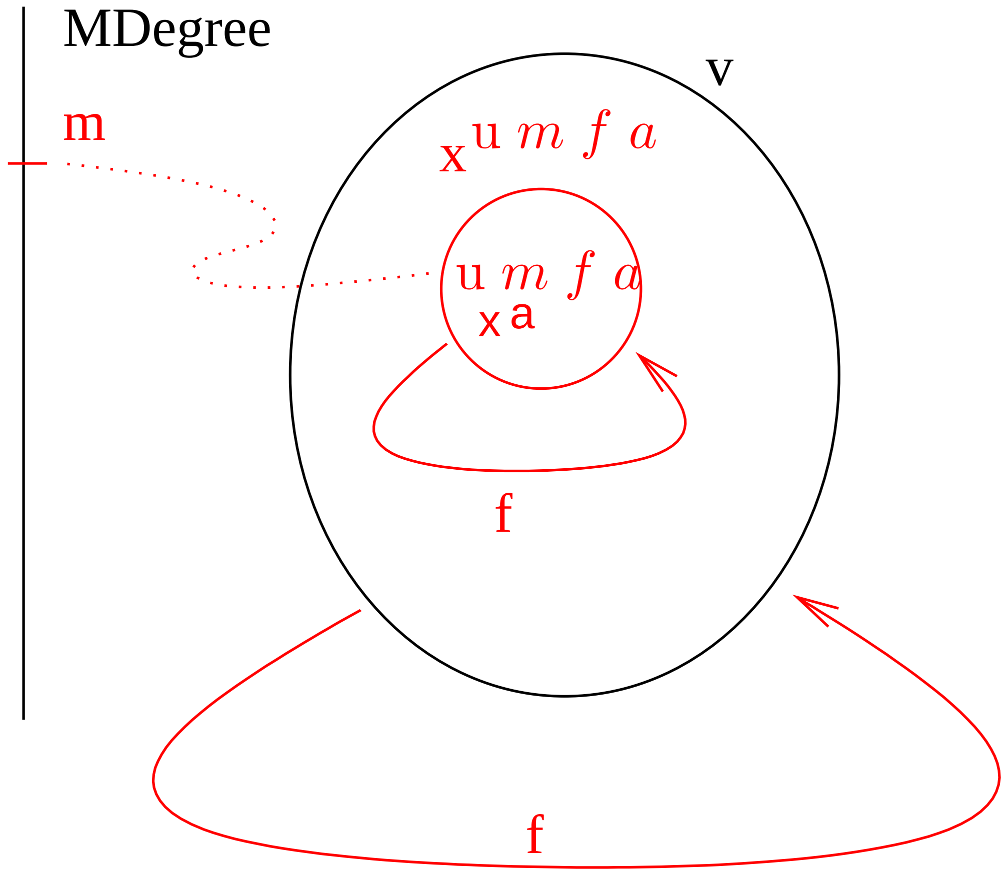
Closure of the Π_3-Refl. Univ.



Closure of the Π_3-Refl. Univ.



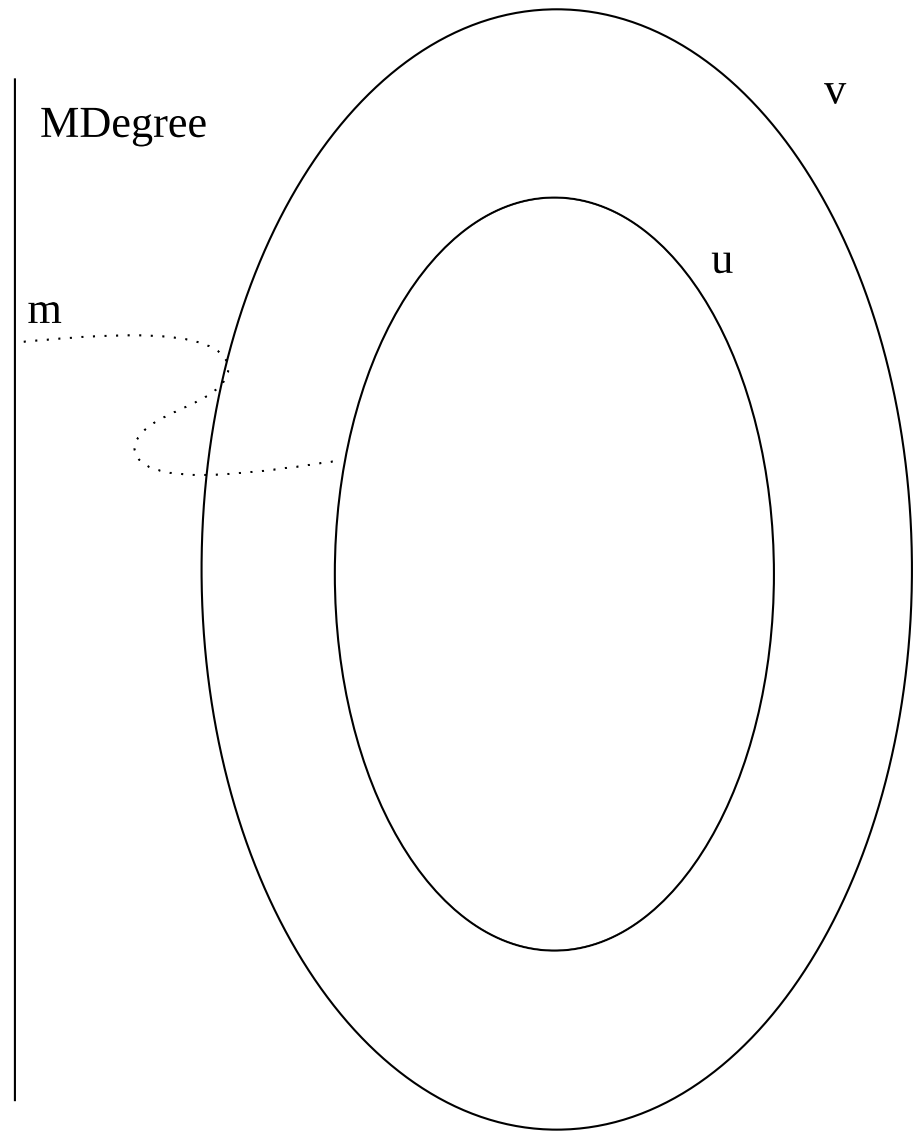
Closure of the Π_3-Refl. Univ.



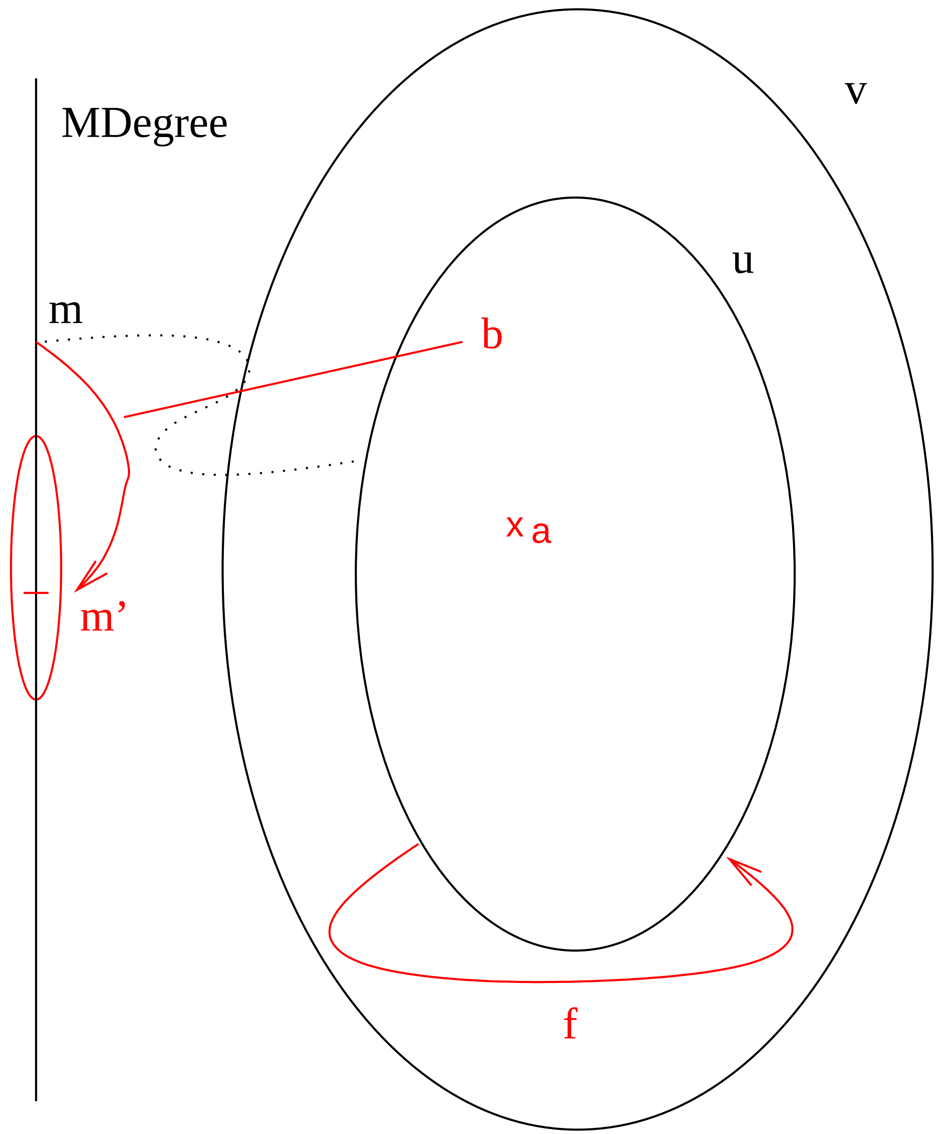
Closure of Universes of Degree m

- Closure of universes of Mahlo degree m:
  - For every $f \in \mathrm{u} \to \mathrm{u}$, $a \in \mathrm{u}$,
  - $b \in \mathrm{u}$, and m' in the family of Mahlo degrees $\mathrm{subdeg}\, m\, b$,
  - there exists a subuniverse of u,
    - closed under a, f,
    - of Mahlo degree m',
    - and represented in u.
- $\mathrm{ML} + \Pi_3 - \mathrm{refl}$ consists of a universe v, Mahlo degrees for v and all rules above.
  - Note that v, the Mahlo degrees and the subuniverses are all defined simultaneously.

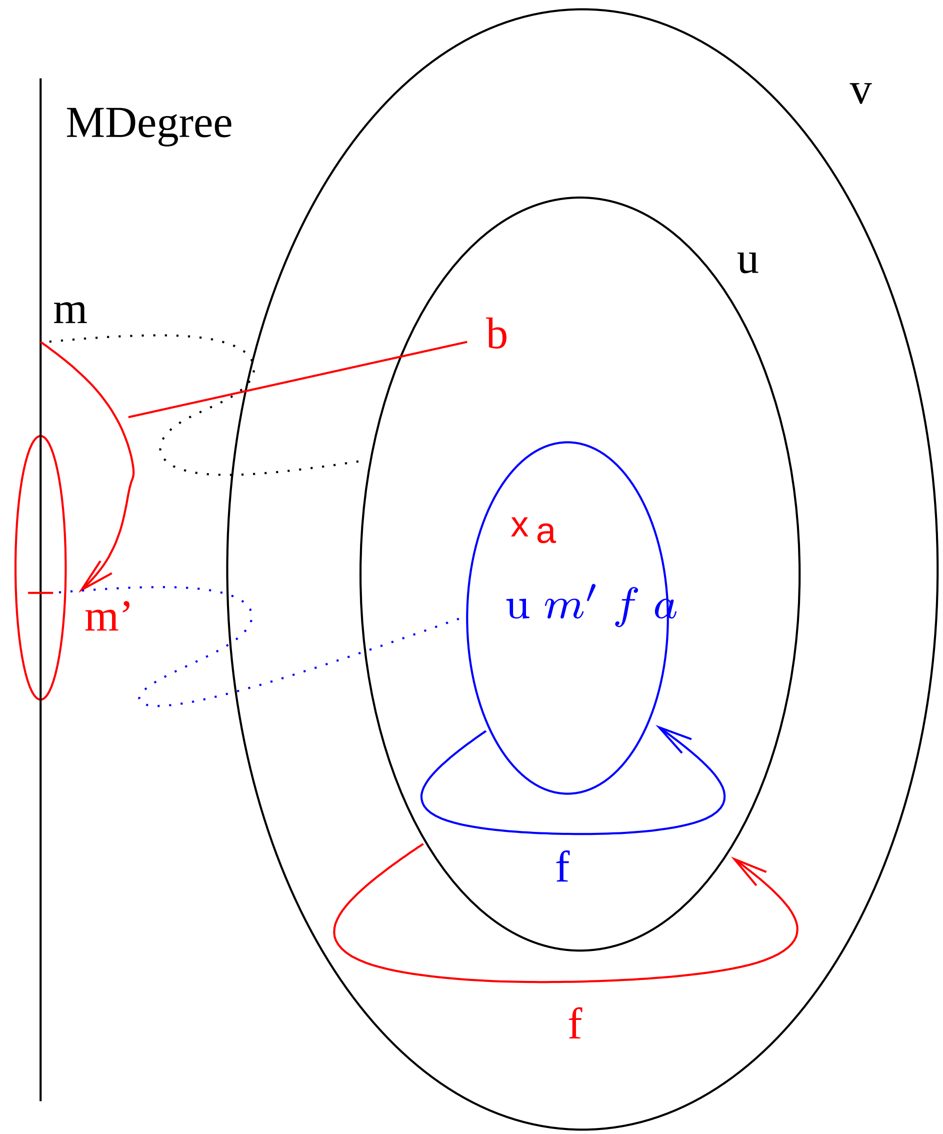
Closure of Universes of Degree m



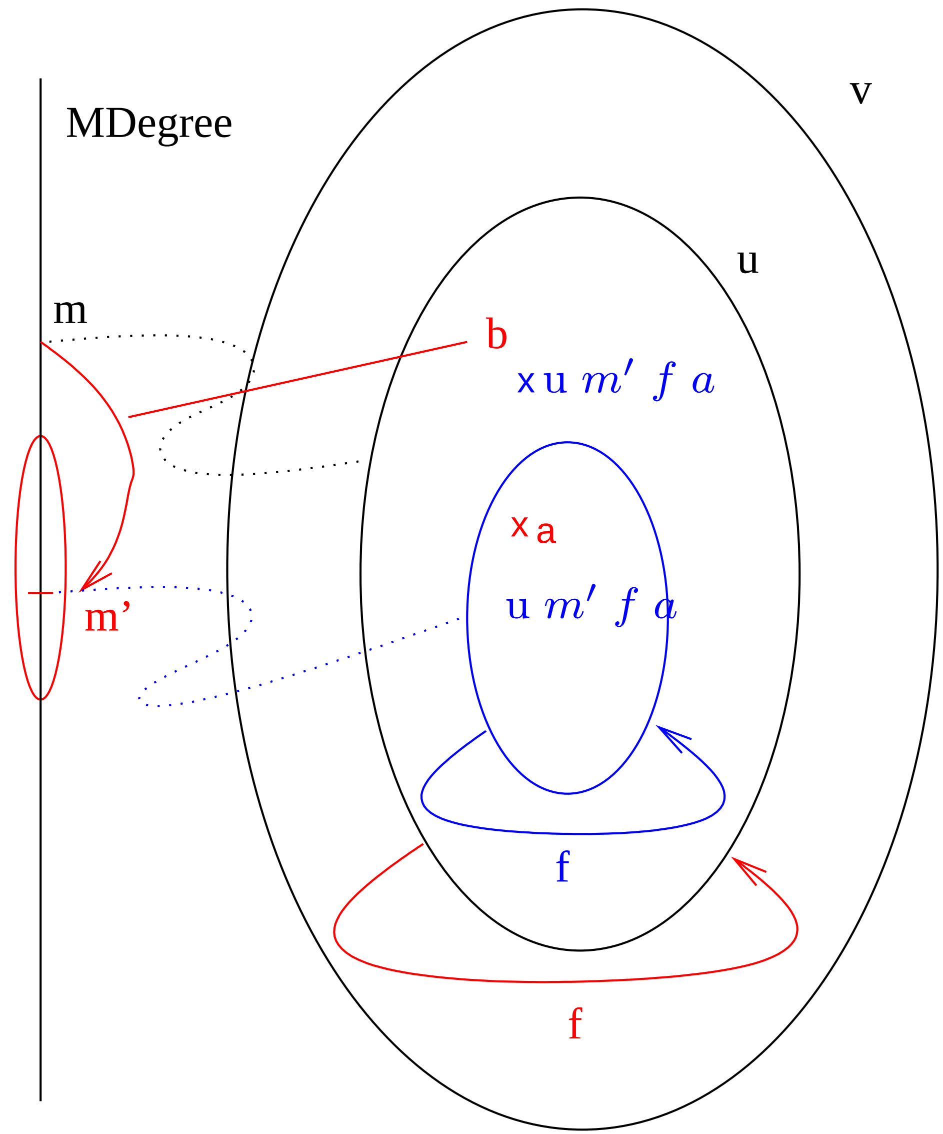
Closure of Universes of Degree m



Closure of Universes of Degree m



Closure of Universes of Degree m



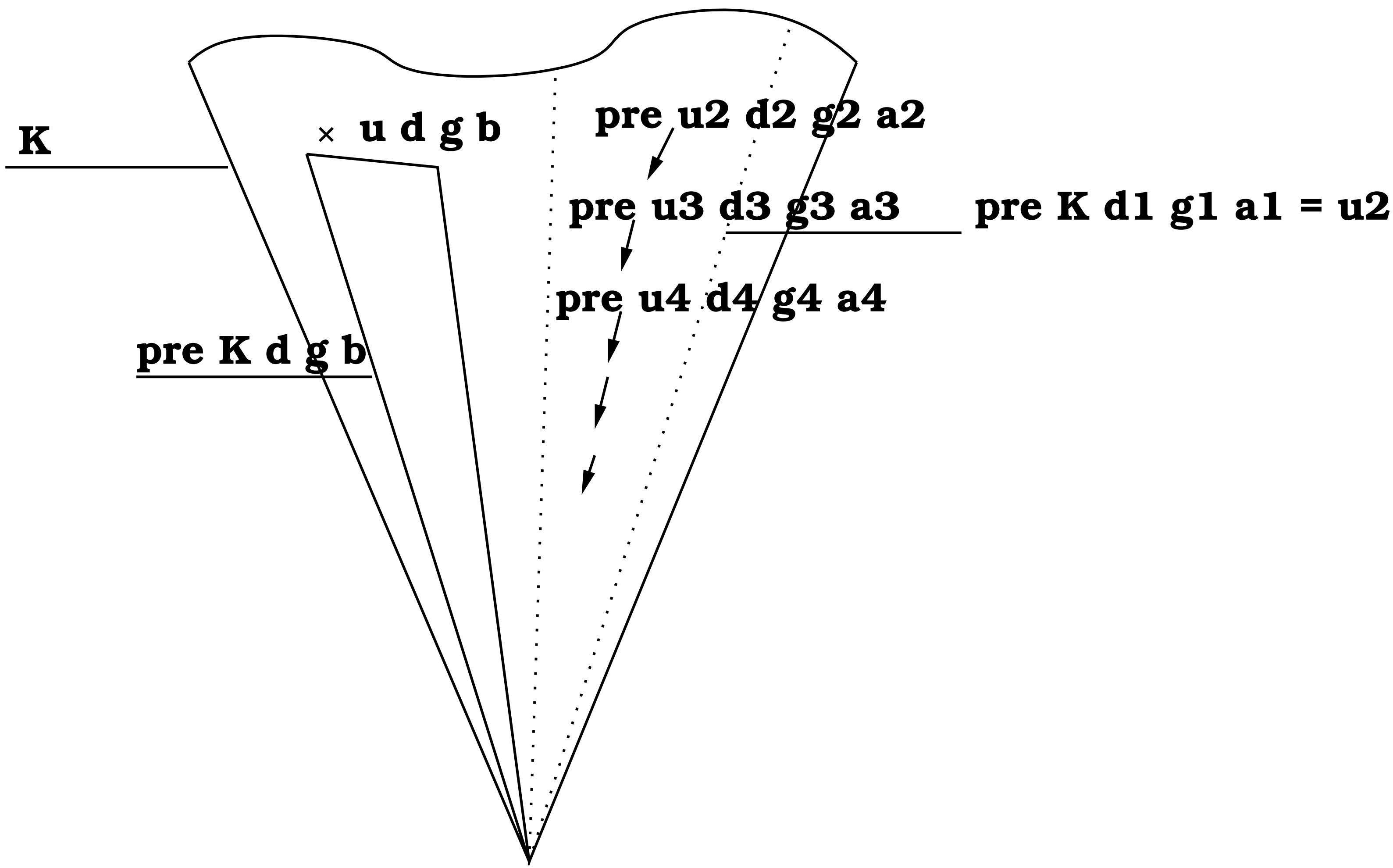
3. Extended Predicative Version

- Remember from the extended predicative Mahlo universe:
  - Let Γ^{Univ} be the operator defining the one-step upward and downward closure of set under universe operations.
  - $\mathrm{RU}(\mathcal{A}, \mathcal{B}, f, a) := (\{a\} \cup f[\mathcal{B}] \cup \Gamma^{\mathrm{Univ}}(\mathcal{B})) \cap \mathcal{A} \subseteq \mathcal{B}$.
  - $\mathrm{Indpt}(\mathcal{A}, \mathcal{B}, f, a) := \{a\} \cup f[\mathcal{B}] \cup \Gamma^{\mathrm{Univ}}(\mathcal{B}) \subseteq \mathcal{A}$.
- Then we had
  - $\mathrm{RU}(u, \mathrm{pre}\ u\ f\ a, f, a)$.
  - $\mathrm{Indpt}(\mathrm{m}, \mathrm{pre}\ \mathrm{m}\ f\ a, f, a) \to \mathrm{u}\ f\ a \in \mathrm{m}$
    $\wedge \mathrm{u}\ f\ a =_{\mathrm{ext}} \mathrm{pre}\ \mathrm{m}\ f\ a\ .$

Idea for Ext. Pred. Π_3-Refl. Univ

- Similarly as before we define $\mathrm{pre}\ u\ d\ f\ a$ by closing it under f, a and Γ^{Univ} relative to u.
- But now $\mathrm{pre}\ u\ d\ f\ a$ is closed under subuniverses of subdegrees of d relative to $\mathrm{pre}\ u\ d\ f\ a$.
- The extended predicative Π_3-reflecting universe K will be closed under all $\mathrm{pre}\ \mathrm{K}\ d\ f\ a$ which are independent of K.

K


K
× u d g b
pre u2 d2 g2 a2
pre u3 d3 g3 a3
pre K d1 g1 a1 = u2
pre u4 d4 g4 a4
pre K d g b

Operations on Degrees

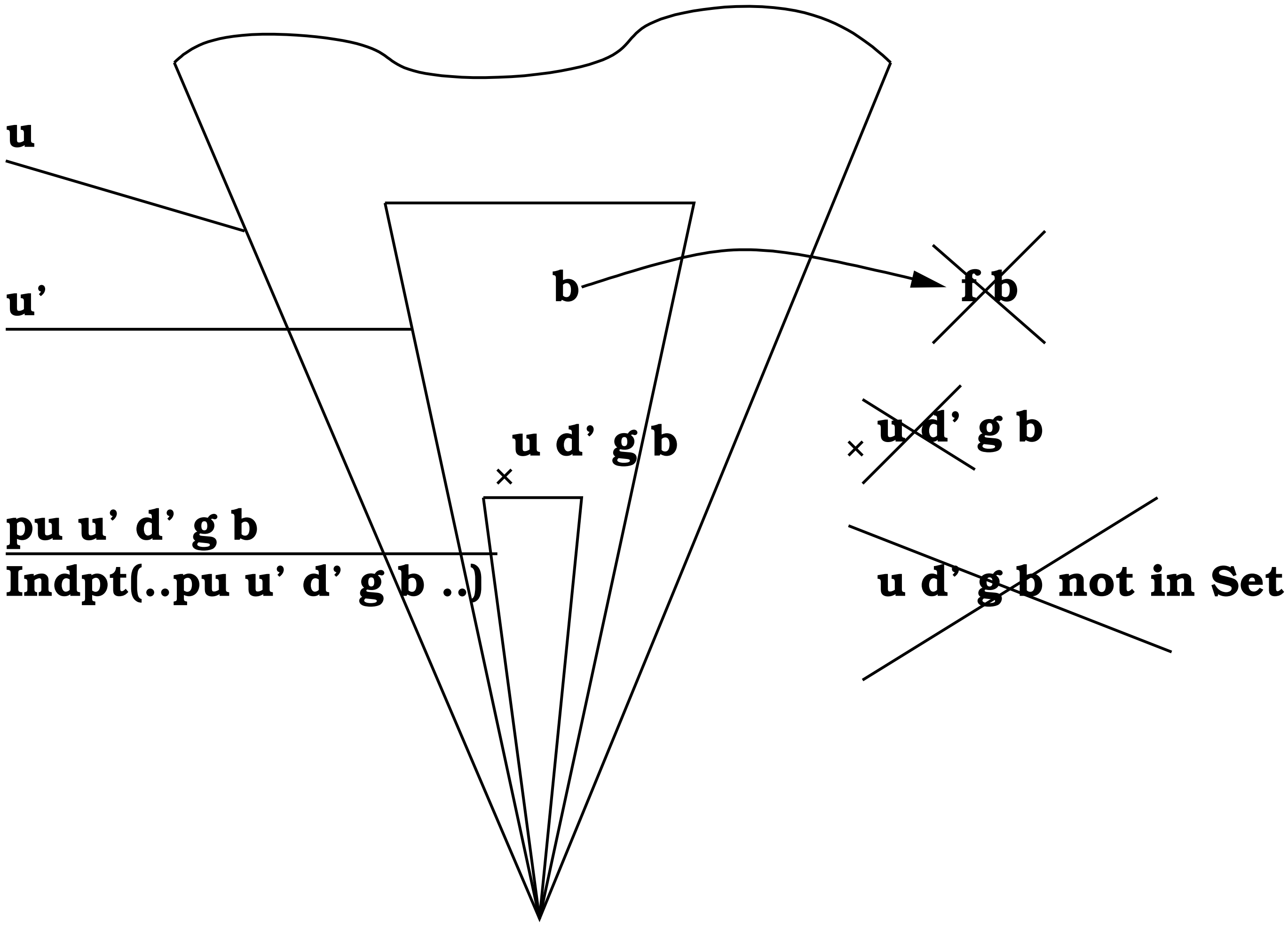
$$
\begin{array}{rcl}
d' \prec^1_{\mathcal{A},\mathcal{B}} d & := & \exists a \in \mathcal{B}.\mathrm{p}_0\,(d\ a) \in \mathcal{A} \wedge \exists b \in \mathrm{p}_0\,(d\ a).\mathrm{p}_1\,(d\ a)\ b = d' \\
\prec_{\mathcal{A},\mathcal{B}}, \preceq_{\mathcal{A},\mathcal{B}} & := & \text{transitive / transitive-reflexive closure of } \prec^1_{\mathcal{A},\mathcal{B}} \\
d \prec_{\mathcal{A}} d' & := & d \prec_{\mathcal{A},\mathcal{A}} d' \text{ etc.}
\end{array}
$$

Closure/Indpt of $\mathrm{pre}\ u\ d\ f\ a$

$$
\begin{aligned}
\mathrm{RU}_{\Pi_3}(\mathcal{A}, v, d, f, a) \quad := \quad & \mathrm{RU}(\mathcal{A}, v, f, a) \\
& \wedge \Gamma^{\mathrm{D}}_{\mathcal{A},d}(v) \cap \mathcal{A} \subseteq v \\
& \wedge \forall d' \prec_{\mathcal{A},v} d. \forall g, b. \\
& \qquad \mathrm{Indpt}_{\Pi_3}(v, \mathrm{pre}\ v\ d'\ g\ b, d', g, b) \\
& \qquad \wedge \mathrm{u}\ d'\ g\ b \in \mathcal{A} \\
& \qquad \rightarrow \mathrm{u}\ d'\ g\ b \in v
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Indpt}_{\Pi_3}(\mathcal{A}, u, d, f, a) \quad := \quad & \mathrm{Indpt}(\mathcal{A}, u, f, a) \wedge \Gamma^{\mathrm{D}}_{\mathcal{A},d}(u) \subseteq \mathcal{A} \wedge \\
& \forall d' \prec_{\mathcal{A},u} d. \forall g, b. \\
& \quad \mathrm{Indpt}(u, \mathrm{pre}\ u\ d'\ g\ b, g, b) \\
& \quad \rightarrow \mathrm{u}\ d'\ g\ b \in \mathcal{A}
\end{aligned}
$$

$\mathbf{Indpt}_{\Pi_3}(\mathbf{u}, \mathrm{v}, \mathbf{d}, \mathbf{f})$



Axioms for Ext. Pred. Π_3

- I. Least pre-universes

(EPΠ_3.1) $u \in \Re_{\Re} \wedge \mathrm{v} = \mathrm{pre}\ u\ d\ f\ a$
$\quad \rightarrow \mathrm{v} \in \Re_{\Re} \wedge \mathrm{RU}_{\Pi_3}(u, \mathrm{v}, d, f, a)$

(EPΠ_3.2) $(u \in \Re_{\Re} \wedge \mathrm{v} = \mathrm{pre}\ u\ d\ f\ a \wedge v' \in \Re_{\Re} \wedge$
$\mathrm{RU}_{\Pi_3}(u, v', d, f, a))$
$\quad \rightarrow \mathrm{v} \subseteq v'$.

(EPΠ_3.3) $(u \in \Re_{\Re} \wedge \mathrm{v} = \mathrm{pre}\ u\ d\ f\ a \wedge \mathcal{A} \subseteq \Re \wedge$
$\quad \mathrm{RU}(u, \mathcal{A}, f, a) \wedge \Gamma^{\mathrm{D}}_{u,d}(\mathcal{A}) \cap u \subseteq \mathcal{A} \wedge$
$\quad \forall d' \prec_{u,\mathcal{A}} d. \forall g, b.$
$\quad \mathrm{Indpt}_{\Pi_3}(\mathcal{A}, \mathrm{pre}\ \mathrm{v}\ d'\ g\ b, d', g, b)$
$\quad \wedge \mathrm{pre}\ \mathrm{v}\ d'\ g\ b \subseteq \mathcal{A} \wedge \mathrm{u}\ d'\ g\ b \in u$
$\quad \rightarrow \mathrm{u}\ d'\ g\ b \in \mathcal{A}))$
$\rightarrow \mathrm{v} \subseteq \mathcal{A}$.

Axioms for Ext. Pred. Π_3

- II. The Π_3-reflecting Universe

$$(\mathrm{EP}\Pi_3.4) \quad \mathrm{K} \in \Re_{\Re} \wedge \Gamma^{\mathrm{Univ}}(\mathrm{K}) \subseteq \mathrm{K} \wedge \\ \forall d, f.\mathrm{Indpt}_{\Pi_3}(\mathrm{K}, \mathrm{pre}\ \mathrm{K}\ d\ f\ a, d, f, a) \\ \rightarrow \mathrm{u}\ d\ f\ a \in \Re \wedge \mathrm{u}\ d\ f\ a \in \mathrm{K} \\ \wedge \mathrm{u}\ d\ f\ a =_{\mathrm{ext}} \mathrm{pre}\ \mathrm{K}\ d\ f\ a$$

$$(\mathrm{EP}\Pi_3.5) \quad (v \in \Re_{\Re} \wedge \Gamma^{\mathrm{Univ}}(v) \subseteq v \wedge \\ \forall d, f.(\mathrm{Indpt}_{\Pi_3}(v, \mathrm{pre}\ v\ d\ f\ a, d, f, a) \\ \rightarrow \mathrm{u}\ d\ f\ a \in v) \\ \rightarrow \mathrm{K} \subseteq v \quad .$$

Axioms for Ext. Pred. Π_3

$$(\mathrm{EP\Pi_3.6}) \qquad \begin{array}{l}(\mathcal{A} \subseteq \Re \wedge \Gamma^{\mathrm{Univ}}(\mathcal{A}) \subseteq \mathcal{A} \wedge \\ \quad \forall d, f.(\mathrm{Indpt}_{\Pi_3}(\mathcal{A}, \mathrm{pre}\ \mathrm{K}\ d\ f\ a, d, f, a) \\ \qquad\qquad \wedge \mathrm{pre}\ \mathrm{K}\ d\ f\ a \subseteq \mathcal{A}) \\ \qquad\qquad \rightarrow \mathrm{u}\ d\ f\ a \in \mathcal{A}) \\ \rightarrow \mathrm{K} \subseteq \mathcal{A}\end{array}$$

Monotonicity Properties

(a) $\mathrm{Indpt}_{\Pi_3}(\mathcal{A}, u, d, f, a) \land \mathcal{A} \subseteq \mathcal{B} \to \mathrm{Indpt}_{\Pi_3}(\mathcal{B}, u, d, f, a)$.

(b) $u \subseteq u' \to \mathrm{pre}\ u\ d\ f\ a \subseteq \mathrm{pre}\ u'\ d\ f\ a$.

(c) $$u \subseteq u' \land \mathrm{Indpt}_{\Pi_3}(u, \mathrm{pre}\ u\ d\ f\ a, d, f, a)$$
$$\to \mathrm{Indpt}_{\Pi_3}(u', \mathrm{pre}\ u\ d\ f\ a, d, f, a)$$
$$\land \mathrm{pre}\ u\ d\ f\ a =_{\mathrm{ext}} \mathrm{pre}\ u'\ d\ f\ a \quad .$$

Relationship between Ind. Principle

(a) $(\text{EP}\Pi_3.1)$, $(\text{EP}\Pi_3.2)$ imply $(\text{EP}\Pi_3.3)$ where $\mathcal{A}$ is a set.

(b) $(\text{EP}\Pi_3.4)$, $(\text{EP}\Pi_3.5)$ imply $(\text{EP}\Pi_3.6)$ where $\mathcal{A}$ is a set.

Interpretation of Mahlo Degrees

$$
\begin{aligned}
&\mathrm{MDegree} := \\
&\quad \{d \mid \Gamma^{\mathrm{D}}_{\mathrm{K},d}(\mathrm{K}) \subseteq \mathrm{K} \\
&\qquad \wedge \forall d' \preceq_{\mathrm{K}} d. \\
&\qquad\quad \forall k \geq 0. \\
&\qquad\quad \forall d_1.f_1, a_1.\mathrm{v}_1 = \mathrm{pre}\ \mathrm{K}\ d_1\ f_1\ a_1 \to \\
&\qquad\quad \forall d_2 \prec_{\mathrm{K}} d_1.\forall f_2, a_2.\mathrm{v}_2 = \mathrm{pre}\ \mathrm{v}_1\ d_2\ f_2\ a_2 \to \\
&\qquad\quad \cdots \\
&\qquad\quad \forall d_k \prec_{\mathrm{K}} d_{k-1}.\forall f_k, a_k.\mathrm{v}_k = \mathrm{pre}\ \mathrm{v}_{k-1}\ d_k; f_k\ a_k \to \\
&\qquad\quad d' \prec_{\mathrm{K}} d_k \to \forall f, a. \\
&\qquad\quad \mathrm{Indpt}(\mathrm{v}_k, \mathrm{pre}\ \mathrm{v}_k\ d'\ f\ a, f, a) \to \mathrm{u}\ d'\ f\ a \in \mathrm{v}_k\} \ .
\end{aligned}
$$

Closure of $\mathrm{MDegree}$

(a) Let d s.t.

- for $a \in \mathrm{K}$ we have $\mathrm{p}_0\ (d\ a) \in \mathrm{K} \wedge \forall x \in \mathrm{p}_0\ (d\ a).(\mathrm{p}_1\ (d\ a))\ x \in \mathrm{MDegree}$.

Then $d \in \mathrm{MDegree}$.

(b) If $d \in \mathrm{MDegree}$, $a \in \mathrm{K}$, then

- $\mathrm{p}_0\ (d\ a) \in \mathrm{K}$
- and for $x \in \mathrm{p}_0\ (d\ a)$ we have $\mathrm{p}_1\ (d\ a)\ x \in \mathrm{MDegree}$.

Definition Univ_d

- $\mathrm{Univ}_d := \{\mathrm{u}\ d\ f\ a \in \mathrm{K} \mid f, a\}$.

Closure Properties

(a) $d \in \mathrm{MDegree} \wedge \mathrm{v} = \mathrm{u}\ d\ f\ a \in \mathrm{Univ}_d$

$\rightarrow \mathrm{v} \subseteq \mathrm{K} \wedge \mathrm{Indpt}_{\Pi_3}(\mathrm{K}, \mathrm{v}, d, f, a)$

$\wedge \mathrm{RU}_{\Pi_3}(\mathrm{K}, \mathrm{v}, d, f, a)$.

Especially $a \in \mathrm{v}$, $f \in \mathrm{v} \rightarrow \mathrm{v}$, $\Gamma^{\mathrm{Univ}}(\mathrm{v}) \subseteq \mathrm{v}$.

(b) Assume $d \in \mathrm{MDegree}$, $f \in \mathrm{K} \rightarrow \mathrm{K}$ and $a \in \mathrm{K}$.
Then $\mathrm{u}\ d\ f\ a \in \mathrm{K}$.

(c) Assume $d \in \mathrm{MDegree}$ and $u \in \mathrm{Univ}_d$, $b \in \mathrm{Fam}(u)$.
Then $\mathrm{p}_0\ (d\ b) \in u$.
Assume furthermore $x \in \mathrm{p}_0\ (d\ b)$, $d' := (\mathrm{p}_1\ (d\ b))\ x$,
$g \in u \rightarrow u$ and $b \in u$.
Then $\mathrm{v} := \mathrm{u}\ d'\ g\ b \in u \cap \mathrm{Univ}_{d'}$ and $b \in \mathrm{v}$ and $g \in \mathrm{v} \rightarrow \mathrm{v}$.

Theorem

- The constructed Π_3-reflecting universe can be interpreted in the extended predicative Π_3-reflecting universe.

Conclusion

- Introduction of an extended predicative Π_3-reflecting universe.
- If complete proof theoretic analysis carried out we get a complete constructive predicative justification of Π_3-reflection.
- It is still open how to obtain a fully predicative version of the Π_3-reflecting universe in MLTT.