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Geometric Aspects of the Effective Topos

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I. The effective topos

Literature:

Martin Hyland – The effective topos, in: Proc. Brouwer Cent. Symp., 1982

Jaap van Oosten – Realizability: an Introduction to its Categorical Side #152 of Studies in Logic, 2008

Notation:

ϕ_n : n-th partial recursive function

$\langle \cdot, \dots, \cdot \rangle$: coding of tuples

If $A, B \subseteq \mathbb{N}$

$$A \Rightarrow B = \{ n \mid \forall a \in A \ \phi_n(a) \in B \}$$

$$A \otimes B = \{ \langle a, b \rangle \mid a \in A, b \in B \}$$

$$A \oplus B = \{0\} \otimes A \cup \{1\} \otimes B$$

An <u>assembly</u> is a pair (X, E) X a set, $E : X \to \mathcal{P}(\mathbb{N})$ s.t. $E(x) \neq \emptyset$, $x \in X$

A <u>map of assemblies</u> $(X, E) \longrightarrow (Y, F)$ is a function $f : X \to Y$ such that

$$\bigcap_{x \in X} \big(E(x) \Rightarrow F(f(x)) \big) \neq \emptyset$$

i.e. for some $n \in \mathbb{N}$ we have:

$$\forall x \in X \ \forall m \in E(x) \ \phi_n(m) \in F(f(x))$$

Say: n <u>tracks</u> f

Have a category **Ass** of assemblies and their maps.

Ass has:
- finite limits
- finite colimits
- a stable regular epi–mono factorization
- locally cartesian closed structure

E.g. $(X,E)\times(Y,F) = (X\times Y,\ (x,y)\mapsto E(x)\otimes F(y))$

$(X,E)+(Y,F) = (X\sqcup Y,\ x\mapsto\{\langle 0,a\rangle \mid a\in E(x)\}$
$y\mapsto\{\langle 1,b\rangle \mid b\in F(y)\}$
$)$

$(Y,F)^{(X,E)} = (\mathrm{Ass}((X,E),(Y,F)),$
$f\mapsto\{n \mid n \text{ tracks } f\})$

Given

$$\begin{array}{ccc} (U,A) & & \\ {\scriptstyle u}\downarrow & & \\ (X,E) & \xrightarrow[f]{} & (Y,F) \end{array}$$

$\Pi_f(u) = (Z,G)$ with

$Z = \{(\alpha,y) \mid \alpha: f^{-1}(y)\to U$ is a partial section of $u\}$

$G(\alpha,y) = \{\langle n,m\rangle \mid n \text{ tracks } \alpha,\ m\in F(y)\}$

(Dependent product)

$(X,E) \xrightarrow{f} (Y,F)$ is a regular epimorphism iff
(f is surjective and)

$$\bigcap_{y \in Y} \Big(F(y) \Rightarrow \bigcup_{f(x)=y} E(x) \Big) \neq \emptyset$$

Logic. Interpret relation symbols as subobjects

If $x_1 : (X_1, E_1), \ldots, x_n : (X_n, E_n)$,

$$[\![R(x_1, \ldots, x_n)]\!] \subseteq (X_1, E_1) \times \ldots \times (X_n, E_n)$$
$$\| \quad$$
$$(X', E')$$

$X' \subseteq X_1 \times \ldots \times X_n$ and the inclusion is tracked

Applying usual constructions of categorical logic yields clauses of Kleene realizability:

The set of subobjects of (X,E) is a Heyting algebra. If $(X_1,E_1), (X_2,E_2) \subseteq (X,E)$ then

$$(X_1,E_1) \cap (X_2,E_2) = (X_1 \cap X_2,\ x \mapsto E_1(x) \otimes E_2(x))$$

$$(X_1,E_1) \to (X_2,E_2) =$$

$$\big(\{ x \in X \mid \exists n\, \forall m\, (m \in E_1(x) \text{ implies } \phi_n(m) \in E_2(x)) \},$$
$$x \mapsto \{ n \mid \forall m\, (m \in E_1(x) \text{ implies } \phi_n(m) \in E_2(x)) \} \big)$$

Can interpret multi-sorted intuitionistic logic without equality in Ass

The effective topos results by 'adding equality' to Ass.

- Consider pairs $((X,E),R)$ with (X,E) an assembly and R a subobject of $(X,E)\times(X,E)$ such that

  Ass $\models$ 'R is an equivalence relation'

- Consider, for such pairs $((X,E),R)$ and $((Y,F),S)$,

  functional relations: subobjects T of $(X,E)\times(Y,F)$ such that

  - $R(x',x)\wedge T(x,y)\wedge S(y,y')\rightarrow T(x',y')$
  - $T(x,y)\wedge T(x,y')\longrightarrow S(y,y')$
  - $E(x)\longrightarrow \exists y\, T(x,y)$

  are true in Ass

The effective topos Eff has as objects $((X,E),R)$ and as arrows: functional relations

Eff is the exact completion of Ass, preserving the regular epimorphisms.

Another formulation:

Objects pairs $(X, \sim)$ with $\sim : X \times X \to P(\mathbb{N})$ such that

$$\bigcap_{x, x'} \left([x \sim x'] \Rightarrow [x' \sim x] \right) \neq \emptyset$$

$$\bigcap_{x, y, z} \left([x \sim y] \otimes [y \sim z] \Rightarrow [x \sim z] \right) \neq \emptyset$$

Arrows Equivalence classes of functions $X \times Y \xrightarrow{F} P(\mathbb{N})$ satisfying

$$\bigcap_{x, y} \left(F(x, y) \Rightarrow ([x \sim x] \otimes [y \sim y]) \right) \neq \emptyset$$

$$\bigcap_{x', x, y, y'} \left([x' \sim x] \otimes F(x, y) \otimes [y \sim y'] \Rightarrow F(x', y') \right) \neq \emptyset$$

$$\bigcap_{x, y, y'} \left(F(x, y) \otimes F(x, y') \Rightarrow [y \sim y'] \right) \neq \emptyset$$

$$\bigcap_{x} \left([x \sim x] \Rightarrow \bigcup_{y} F(x, y) \right) \neq \emptyset$$

Two such $F, G : X \times Y \to P(\mathbb{N})$ are equivalent if

$$\bigcap_{x, y} \left(F(x, y) \Rightarrow G(x, y) \right) \neq \emptyset$$

In special cases, every arrow $(X,\sim) \xrightarrow{F} (Y,\sim)$ is given by a function $f: X \to Y$ satisfying

$$\bigcap_{x,x'} \left([x \sim x'] \Rightarrow [f(x) \sim f(x')] \right)$$

Then F is equivalent to

$$(x,y) \longmapsto [x \sim x] \otimes [f(x) \sim y]$$

Such cases are:

1) If $[x \sim x]$ is a singleton for each x

2) If $(Y,\sim)$ is a power object $\Omega^{(Z,\sim)}$

$\Omega = (\mathcal{P}(\mathbb{N}), \sim)$ where

$[A \sim B] = \{ \langle a,b \rangle \mid a \in A \Rightarrow B,\ b \in B \Rightarrow A \}$

3) If $(Y,\sim)$ satisfies

$[y \sim y'] = \emptyset$ if $y \neq y'$

From 3): have full embedding $\mathrm{Ass} \longrightarrow \mathrm{Eff}$

$$(X,E) \longmapsto (X,\sim) \text{ with}$$

$$[x \sim x'] = \begin{cases} E(x) & \text{if } x = x' \\ \emptyset & \text{else} \end{cases}$$

Also: full embedding $\nabla : \mathrm{Set} \longrightarrow \mathrm{Ass}$ and by composition $\nabla : \mathrm{Set} \longrightarrow \mathrm{Eff}$

$\nabla(X) = (X,\sim)$ with $[x \sim x'] = \begin{cases} \mathbb{N} & x = x' \\ \emptyset & \text{else} \end{cases}$

$\Omega = (\mathcal{P}(\mathbb{N}), \sim)$

$$A \sim B = (A \Rightarrow B) \otimes (B \Rightarrow A)$$

with 'true': $1 \longrightarrow \Omega$ given by $\top(*) = \mathbb{N}$

$(X, \sim) \xrightarrow{F} (Y, \sim)$ is mono iff

$$\bigcap_{x, x', y} \left(F(x, y) \otimes F(x', y) \Rightarrow [x \sim x'] \right) \neq \emptyset$$

and epi iff

$$\bigcap_{y} \left([y \sim y] \Rightarrow \bigcup_{x} F(x, y) \right) \neq \emptyset$$

If F is mono, its characteristic function $\chi_F : (Y, \sim) \longrightarrow \Omega$ is given by the function

$$y \longmapsto \bigcup_{x} F(x, y)$$

Characterizing assemblies: t.f.a.e. for $(X, \sim)$

a) $(X, \sim)$ is isomorphic to an assembly

b) there is a monomorphism $(X, \sim) \longrightarrow \nabla(Y)$

c) $\mathrm{Eff} \models \forall x, x' : (X, \sim) \quad \neg\neg(x = x') \longrightarrow x = x'$

Objects of form $\nabla(X)$: called <u>sheaves</u>.

For every $(X, \sim)$ there is the <u>associated sheaf</u> $a(X, \sim)$:

Let $\Gamma(X, \sim) = \mathrm{Eff}(1, (X, \sim))$

$$\cong \{x \mid [x \sim x] \neq \emptyset\}/_{\approx} \qquad x \approx x' \text{ if } [x \sim x'] \neq \emptyset$$

$$a(X, \sim) = \nabla\Gamma(X, \sim)$$

If R is a subobject of $(X,\sim)$ then

$$\mathrm{Eff} \models \forall x:(X,\sim)\ \neg\neg R(x) \to R(x)$$

('R is $\neg\neg$-stable')

if and only if

$$\begin{tikzcd} R \arrow[r] \arrow[d] & aR \arrow[d] \\ (X,\sim) \arrow[r] & a(X,\sim) \end{tikzcd}$$

is a pullback diagram.

Natural numbers:

The assembly $N = (\mathbb{N}, n \mapsto \{n\})$, together with $0 \in \mathbb{N}$, '$+1$': $\mathbb{N} \to \mathbb{N}$ is a <u>natural numbers object</u> in Ass (hence in Eff):

initial diagram of form $1 \xrightarrow{x} X \xrightarrow{f} X$

By the exponential structure of Ass, we have

$N^N = (T, E)$ where T = set of total rec. functions
$E(f) = \{n \mid f = \phi_n\}$

$N^{(N^N)} = (U, E)$ where U is the set of effective operations
$E(\phi)$ = set of indices for ϕ

We get:

The finite type structure over N in Eff is isomorphic to the structure of 'hereditarily effective operations' (Troelstra)

Note: all these objects are assemblies of form (X, E) s.t. $E(x) \cap E(x') = \emptyset$ if $x \neq x'$.

Hyland called these: effective objects
Scott " " : modest sets

More generally, have <u>discrete objects</u>:
t.f.a.e. for object $(X, \sim)$

a) $(X, \sim)$ is isomorphic to some object $(X', \sim)$ for which $[x \sim x] \cap [y \sim y] = \emptyset$ if $x \neq y$

b) There is a subobject A of N and an epi $A \twoheadrightarrow (X, \sim)$

c) The diagonal $(X, \sim) \longrightarrow (X, \sim)^{\nabla(2)}$ is an isomorphism

See: 'effective objects' = 'discrete assemblies'

Think of 'discrete' as totally disconnected

Other extreme: uniform objects
t.f.a.e. for $(X, \sim)$

a) There is an epi $\nabla Y \twoheadrightarrow (X, \sim)$

b) $(X, \sim)$ is isomorphic to some object $(X', \sim)$ for which $\bigcap_{x \in X'} [x \sim x] \neq \emptyset$

Can see:

Ω is uniform (if $\phi_i = id$, then $\langle i,i\rangle \in \bigcap_A [A \sim A]$)

Fact: every power object $\Omega^{(X,\sim)}$ is uniform

Fact ('Uniformity Principle'):

If $R \rightarrowtail (U,\sim) \times (D,\sim)$ ~~[illegible]~~ is a total relation from a uniform $(U,\sim)$ to a discrete $(D,\sim)$, then R contains a constant function

In particular, uniform objects are 'indecomposable': cannot be written as nontrivial sum of subobjects.

But: also discrete objects may be indecomposable.

Example. If $K = \{n \mid \phi_n(n) \text{ is defined}\}$

$$\Sigma = (\{0,1\}, E) \qquad E(0) = K$$
$$E(1) = \mathbb{N} - K$$

Then $\Sigma^N \simeq (RE, F)$ RE set of r.e. subsets of $\mathbb{N}$

$$F(A) = \{e \mid A = W_e\}$$

Then Σ^N discrete, and indecomposable (Rice theorem)

Also, the object R of real numbers, is discrete but indecomposable (by Brouwer's 'theorem' in Eff, and R is topologically connected)

For discussing notions 'uniform' and 'discrete' in parameters, define when a map $(Y,\sim) \to (X,\sim)$ is uniform or discrete.

Def. a) $(Y,\sim) \to (X,\sim)$ is uniform iff there is a comm. diagram

$$\begin{tikzcd}
(Y,\sim) \arrow[dr, "f"'] & (Z,\sim) \arrow[l, two heads, "e"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & \nabla W \arrow[d] \\
& (X,\sim) \arrow[r] & a(X,\sim)
\end{tikzcd}$$

with $(*)$ a pullback, and e epi

(There is an epi from a sheaf to f in the slice topos $\mathrm{Eff}/(X,\sim)$)

b) $(Y,\sim) \xrightarrow{f} (X,\sim)$ is discrete iff there is a comm. diagram

$$\begin{tikzcd}
N \times (X,\sim) \arrow[dr, "\bar{\pi}"'] & (A,\sim) \arrow[l, tail, "m"'] \arrow[r, two heads, "e"] \arrow[d] & (Y,\sim) \arrow[dl, "f"] \\
& (X,\sim) &
\end{tikzcd}$$

with m mono and e epi

(There is an epi from a subobject of N to f in the slice $\mathrm{Eff}/(X,\sim)$)

A. Carboni has investigated an analogy in Eff of the following:

The adjunction $\mathbb{S} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbb{CH}$ of Stone Spaces into Compact Hausdorff spaces gives a factorization system in $\mathbb{CH}$:

Every map in $\mathcal{CH}$ factors as 'monotone' (connected fibers) followed by 'light' (totally disconnected fibers)

The analogue in Eff gives something like:

$$\begin{tikzcd}
(X,\sim) \arrow[rr, "f"] \arrow[dr, "\text{'uniform'}"'] & & (Y,\sim) \\
& (Z,\sim) \arrow[ur, "\text{'discrete'}"'] &
\end{tikzcd}$$

In this talk, I am interested in essentially different factorization: of form

$$\longrightarrow \quad \xrightarrow{\underline{\text{uniform}}}$$

The inspiration comes from the notion of a 'closed model structure' (D. Quillen)

II. Closed Model Categories

Literature: e.g.

H. J. Baues, Algebraic Homotopy (CUP 1989)

M. Hovey, Model Categories (? ±1999)

Suppose $f: A \to B$, $g: C \to D$ are arrows in a category. Then f has the <u>left lifting property</u> (llp) wrt g, and g has the <u>right lifting property</u> wrt f, if

$$\forall \begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd} \quad \exists \begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur] & D \end{tikzcd}$$

Example: an object X is projective iff the map $0 \to X$ has the llp wrt all epis.

Say also for classes $\mathcal{C}, \mathcal{D}$ of arrows: $\mathcal{C}$ has llp wrt $\mathcal{D}$, if every $f \in \mathcal{C}$ has llp wrt every $g \in \mathcal{D}$.

<u>Def.</u> A closed model structure on a category consists of 3 classes of maps:

$\mathcal{C}$ (cofibrations)

$\mathcal{F}$ (fibrations)

$\mathcal{W}$ (weak equivalences)

such that the following conditions hold:

a) Given $\xrightarrow{f}\xrightarrow{g}$, whenever 2 out of $\{f, g, gf\}$ are in $\mathcal{W}$, then so is the third

b) $\mathcal{C}, \mathcal{F}, \mathcal{W}$ are closed under retracts

c) $\mathcal{C}$ has llp wrt $\mathcal{W} \cap \mathcal{F}$

$\mathcal{C} \cap \mathcal{W}$ has llp wrt $\mathcal{F}$

d) Every arrow f can be factored:

i) as $f = gi$, $g \in \mathcal{F}$, $i \in \mathcal{C} \cap \mathcal{W}$

ii) as $f = hk$, $h \in \mathcal{F} \cap \mathcal{W}$, $k \in \mathcal{C}$

From definition it follows that, given $\mathcal{W}$, the classes $\mathcal{C}$ and $\mathcal{F}$ determine each other:

e.g. $\mathcal{C} = \{f \mid f \text{ has llp wrt } \mathcal{W} \cap \mathcal{F}\}$

<u>Example</u> In (compactly generated) topological spaces:

$\mathcal{F}$ = Serre fibrations

$\mathcal{W}$ = weak homotopy equivalences

Given a model structure on a category, have 'homotopy relation' between arrows:

If B, X are objects, define

- a cilinder for B is a factorization

$$B + B \xrightarrow{\in \mathcal{C}} B' \xrightarrow{\in \mathcal{F} \cap \mathcal{W}} B$$

- a path object for X is a factorization

$$X \xrightarrow{\in \mathcal{C} \cap \mathcal{W}} X' \xrightarrow{\in \mathcal{F}} X \times X$$

For $f, g : B \longrightarrow X$ say:

$f \sim_{\ell} g$ (f is left homotopic to g) if

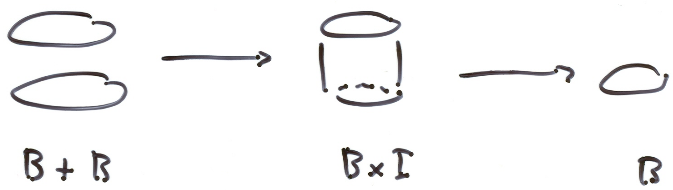
$\begin{bmatrix} f \\ g \end{bmatrix} : B + B \longrightarrow X$ factors through B'

$f \sim_r g$ (f is right homotopic to g) if

$\langle f, g \rangle : B \longrightarrow X \times X$ factors through X'

For 'good' B, X these relations agree (and don't depend on choice of cilinder or path object)

In Top,



$$X' = X^I \quad (I = [0,1])$$

Paradigmatic example: Simplicial Sets

Δ: category of finite, nonempty ordinals and $\leq$-preserving functions

Sset (category of simplicial sets): $Set^{\Delta^{op}}$

Simplicial sets are important in

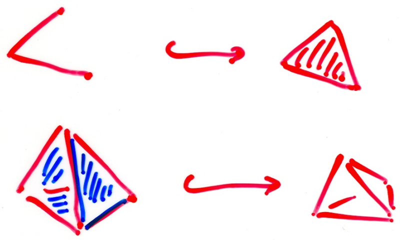
a) Topology: simplicial set is 'glueing instructions' for pasting together basic 'simplices'

to a space

('Geometric realization': $Sset \xrightarrow{R} Top$)

Have closed model structure on Sset, for which

- fibrations have rlp wrt 'horn inclusions'



(take away interior and 1 face)

- weak equivalences are maps f for which R(f) is w.e. in Top

b) Category Theory

Simplicial set X looks like:

$$\begin{tikzcd} X(0) \arrow[r, "s_0"] & X(1) \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left=1] \arrow[r, shift left=3] & X(2) \arrow[l, shift right=4] \arrow[l, "d_2"'] \arrow[l, shift left=4] & \cdots \end{tikzcd}$$

i.e. like a 'category'. Particular ssets model 'higher categories': the quasi-categories of A. Joyal and J. Lurie

There is closed model structure on Sset such that the ssets X for which $X \to 1$ is a fibration, are quasi-categories

c) Logic

Ssets is a topos, and 'classifies linear orders with end points'

Conclusion: Closed model structures are important for theories about some notion of 'equivalence' (like homotopy, eq. of cats.)

Recently, research has started to use closed model structures to model 'identity types' in intensional M-L type theory (Martin-Löf): Awodey-Warren, Richard Garner, Gambino

Back to Eff.

Recall uniform maps $(Y,\sim) \longrightarrow (X,\sim)$:

$$\begin{tikzcd} (Y,\sim) & (A,\sim) \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \nabla W \arrow[d] \\ & (X,\sim) \arrow[r] & a(X,\sim) \arrow[from=1-1, to=2-2] \end{tikzcd}$$

In my book, have following characterization of uniform maps:

A map $F: (Y,\sim) \longrightarrow (X,\sim)$ is uniform iff there are numbers a and b such that for every $x \in X$, $y \in Y$, $n \in [x \sim x]$, $m \in F(y,x)$, there is $y' \in Y$ such that $\phi_a(n) \in F(y',x)$ and $\phi_b(n,m) \in [y' \sim y]$

It turns out that this is a right lifting property.

Call an assembly (X,E) <u>simple</u> if $E(x)$ is always a singleton

Call a map between simple assemblies <u>simple</u> if $f: (X,E) \longrightarrow (Y,E)$ is a bijective function

Then:

<u>Theorem</u> i) A map $F: (Y,\sim) \longrightarrow (X,\sim)$ is uniform iff F has the rlp wrt all simple maps of simple assemblies

ii) F ~~has~~ is uniform and epi iff F has the rlp wrt all monos between simple assemblies

This suggests:

There is, maybe, a closed model structure on Eff for which the fibrations are the (epimorphic) uniform maps.

In fact, Bauer presents a weakening of the closed model structure axioms, still useful.

Def. A *fibration category* has 2 classes W and $\mathcal{F}$ (weak eq. and fibrations). Satisfying:

(F1) $\mathcal{F}$ closed under composition; W satisfies 2-out-of-3

(F2) Given

$$\begin{tikzcd} & A \arrow[d, "f"] \\ Y \arrow[r, "i"'] & B \end{tikzcd}$$

with $f \in \mathcal{F}$, the pullback

$$\begin{tikzcd} Y \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\ Y \arrow[r, "i"'] & B \end{tikzcd}$$

exists and $\pi_2 \in \mathcal{F}$. Moreover:

a) if $i \in W$, $\pi_1 \in W$

b) if $f \in W$, $\pi_2 \in W$

(F3) Every arrow $f: A \to B$ factors as $A \xrightarrow{g} W \xrightarrow{i} B$ with $g \in W$, $i \in F$

(F4) Call Q <u>cofibrant</u> if every $f: P \to Q$ with $f \in F \cap W$, has a section. Then: for every X there is $a: QX \to X$ with $a \in W \cap F$ and QX cofibrant

<u>Prop 1.</u> In Ass, define

$W = \{ f: (X,E) \to (X',E') \mid f \text{ is bijective} \}$

$\mathcal{F} = \{ f: (X,E) \to (Y,E') \mid (X,E) \cong (X, x \mapsto E'(f(x))) \}$

Then $(\mathrm{Ass}, W, \mathcal{F})$ is a fibration category